This leaves us with an obvious follow-up question: Are the weak interactions anomalous? Since the $\mathrm{SU}(2)_{\text{weak}}$ gauge group of the Standard Model only couples to left-handed fields, it seems very dangerous. To answer this question, we need the generalization of the above results to non-Abelian currents. But first we repeat the chiral anomaly calculation using a different technique.

30.3 Chiral anomaly from the integral measure

In the previous section, we calculated the chiral anomaly through Feynman diagrams. In the massless case, this calculation was very subtle and involved a careful choice of momentum in a loop integral. A more direct connection between the anomaly and the violation of a symmetry uses the path integral. The intuitive idea, due to Kazuo Fujikawa, is that anomalies arise when there are symmetries of the action that are not symmetries of the functional measure in the path integral.

To begin, we quickly review the path-integral proof of current conservation in the quantum theory from Section 14.5. We start with

$$\langle \mathcal{O}(x_1,\dots,x_n)\rangle = \frac{1}{Z[0]}\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left[i\int d^4x\, i\bar{\psi}\,\partial\!\!\!/\psi\right]\mathcal{O}(x_1,\dots,x_n), \tag{30.54}$$

where $\mathcal{O}(x_1,\dots,x_n)$ is some gauge-invariant operator. For example, you can think of $\mathcal{O} = J^\mu(y)J^\nu(z)$. This action is invariant under the global symmetries $\psi \to e^{i\alpha}\psi$ and $\psi \to e^{i\beta\gamma_5}\psi$. To derive current conservation for the vector symmetry, we redefine $\psi(x) \to e^{i\alpha(x)}\psi(x)$, with α now a function of x. The measure is invariant under this change of variables (we will confirm this in a moment) and $\mathcal{O}(x_1,\dots,x_n)$ is invariant, but $\bar{\psi}\partial\!\!\!/\psi \to \bar{\psi}\partial\!\!\!/\psi + i\,\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha$. Since the path integral integrates over all field configurations, it is invariant under any field redefinition, thus the remaining term proportional to α must vanish. Expanding to first order in α and integrating by parts, we find

$$0 = \frac{1}{Z[0]}\int d^4z\,\alpha(z)\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left[i\int d^4x\, i\bar{\psi}\,\partial\!\!\!/\psi\right]\frac{\partial}{\partial z^\mu}\Big[\,\bar{\psi}(z)\gamma^\mu\psi(z)\Big]\mathcal{O}(x_1,\dots,x_n)\,. \tag{30.55}$$

Since this holds for all $\alpha(z)$, we must have

$$\partial_\mu\langle J^\mu(x)\mathcal{O}(x_1,\dots,x_n)\rangle = 0. \tag{30.56}$$

The only part of the above derivation that changes when we consider an axial rotation $\psi \to e^{i\beta(x)\gamma_5}\psi$ is that the path integral measure is no longer invariant.

To see how the measure changes, consider a general linear transformation $\psi(x) \to \Delta(x)\psi(x)$ and $\bar{\psi}(x) \to \bar{\psi}(x)\Delta_c(x)$ which generates a Jacobian factor:

$$\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \to [\mathcal{J}_c\,\mathcal{J}]^{-1}\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi. \tag{30.57}$$

The Jacobians $\mathcal{J} = \det\Delta$ and $\mathcal{J}_c = \det\Delta_c$ appear to negative powers because the transformed variables are fermionic (see Section 14.6). To make sense out of $\mathcal{J}$ we write

$$\mathcal{J} = \det\Delta = \exp\,\operatorname{tr}\,\ln\Delta = \exp\left[\int d^4x\,\langle x|\,\operatorname{Tr}\ln\Delta(x)\,|x\rangle\right], \tag{30.58}$$

where the tr sums over all the eigenvalues of Δ and the Tr is a Dirac trace. For the sum over positions, we have introduced a one-particle Hilbert space $\{|x\rangle\}$.[2] For example, consider a non-chiral transformation with $\Delta(x) = e^{i\alpha(x)}$ and $\Delta_c(x) = e^{-i\alpha(x)}$. Then,

$$\mathcal{J} = \mathcal{J}_c^\dagger = \exp\left[4i\int d^4x\delta^4(x-x)\,\alpha(x)\right]. \tag{30.59}$$

where $\langle x|y\rangle = \delta^4(x-y)$ has been used. Despite the infinite $\delta^4(0)$ factor, $\mathcal{J}_c\mathcal{J} = 1$ since the integrand is real and so the measure is invariant. In contrast, for the axial transformation $\Delta(x) = \Delta_c(x) = e^{i\beta(x)\gamma^5}$ and

$$\mathcal{J} = \mathcal{J}_c = \exp\left[i\int d^4x\delta^4(x-x)\,\beta(x)\mathrm{Tr}[\gamma_5]\right], \tag{30.60}$$

Since $\delta^4(0)\mathrm{Tr}[\gamma_5]$ gives infinity times zero, $\mathcal{J}_c\mathcal{J}$ is now undefined.

In full QED, the situation is similar. The QED path integral is

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A\,\exp\left[i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}\not{D}\psi\right)\right]. \tag{30.61}$$

The action is still invariant under the global symmetries $\psi \to e^{i\alpha}\psi$ and $\psi \to e^{i\beta\gamma_5}\psi$ with A_μ unchanged. Under the local axial transformation, A_μ is invariant, so its transformation does not contribute to the Jacobian.

To regulate the undefined product in Eq. (30.60), let us first write $\mathcal{J}$ as:

$$\mathcal{J} = \exp\left(i\int d^4x\,\mathrm{Tr}[\langle x|\beta(\hat{x})\gamma_5|x\rangle]\right), \tag{30.62}$$

Now, we regulate the divergence in a gauge-invariant manner by introducing an exponential regulator of the form $\exp(-\hat{\not{\Pi}}^2/\Lambda^2)$, where $\hat{\not{\Pi}} = \hat{\not{p}} - e\not{A}(\hat{x})$, Λ is some UV cutoff and $\hat{p}$ is the operator conjugate to $\hat{x}$ in the one-particle Hilbert space. The relation $\not{D}^2 = D_\mu^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}$, from Eq. (10.106), implies

$$\hat{\not{\Pi}}^2 = \hat{\Pi}^2 - \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}(\hat{x}), \tag{30.63}$$

so that

$$\begin{aligned}\mathrm{Tr}[\langle x|\beta(\hat{x})\gamma^5|x\rangle] &= \lim_{\Lambda\to\infty}\mathrm{Tr}\left[\langle x|\beta(\hat{x})\,\gamma^5 e^{\hat{\not{\Pi}}^2/\Lambda^2}|x\rangle\right]\\ &= \lim_{\Lambda\to\infty}\beta(x)\langle x|\mathrm{Tr}\left[\gamma^5\exp\left(\frac{(\hat{p}-eA(\hat{x}))^2 - \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}}{\Lambda^2}\right)\right]|x\rangle.\end{aligned} \tag{30.64}$$

Now, the trace of a product of γ-matrices with one γ^5 vanishes unless there are at least four γ-matrices in the product. Thus, the leading term in the expansion of the exponential

[2] To interpret this expression, we do not need a physical interpretation of the one-particle Hilbert space – we just want to use the mathematical tricks we learned in quantum mechanics to write the sum over positions in a suggestive form. There is in fact a beautiful interpretation of one-particle Hilbert spaces like this in quantum field theory, to which much of Chapter 33 is devoted.

is of order $\frac{1}{\Lambda^4}$. Using the identity $\frac{1}{2}\{\sigma^{\mu\nu}, \sigma^{\alpha\beta}\} = g^{\mu\alpha}g^{\nu\beta}\mathbb{1} - g^{\nu\alpha}g^{\mu\beta}\mathbb{1} + i\gamma^5\varepsilon^{\mu\nu\alpha\beta}$, where $\mathbb{1}$ is the identity matrix with Dirac indices, we can derive that

$$(\sigma_{\mu\nu}F^{\mu\nu})^2 = 2F_{\mu\nu}^2\mathbb{1} + i\gamma^5\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}, \tag{30.65}$$

which leads to

$$\begin{aligned}&\mathrm{Tr}[\langle x|i\beta(\hat{x})\gamma^5|x\rangle]\\ &\quad = -\frac{e^2}{2}\beta(x)\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)\lim_{\Lambda\to\infty}\left[\frac{1}{\Lambda^4}\langle x|e^{(\hat{p}-eA)^2/\Lambda^2}|x\rangle + \mathcal{O}\left(\frac{1}{\Lambda^5}\right)\right].\end{aligned} \tag{30.66}$$

To extract the contribution leading in e, we can set $A = 0$ in the exponent. Next insert $\mathbb{1} = (2\pi)^{-4}\int d^4k|k\rangle\langle k|$ with $\hat{p}|k\rangle = k|k\rangle$ to get

$$\frac{1}{\Lambda^4}\langle x|e^{\hat{p}^2/\Lambda^2}|x\rangle = \frac{1}{\Lambda^4}\int\frac{d^4k}{(2\pi)^4}e^{k^2/\Lambda^2} = \frac{i}{\Lambda^4}\int\frac{d^4k_E}{(2\pi)^4}e^{-k_E^2/\Lambda^2} = \frac{i}{16\pi^2}. \tag{30.67}$$

Thus, we find a finite answer as $\Lambda \to \infty$:

$$\mathcal{J} = \exp\left[-i\int d^4x\left(\beta(x)\frac{e^2}{32\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)\right)\right]. \tag{30.68}$$

Note that, if we had used $e^{-\hat{p}^2/\Lambda^2}$ or $e^{-\Pi^2/\Lambda^2}$, the singularity would not have been regulated and the Jacobian would still be undefined.

The result is that under an axial transformation

$$\begin{aligned}&\int\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A\exp\left[i\int d^4x\,\mathcal{L}_{\mathrm{QED}}\right]\\ &\to\int\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A\exp\left[i\int d^4x\left(\mathcal{L}_{\mathrm{QED}} - J_\mu^5\partial_\mu\beta + \beta\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right)\right].\end{aligned} \tag{30.69}$$

Thus, the Schwinger–Dyson equation in Eq. (30.56) becomes

$$\partial_\mu\langle J^{5\mu}(x)\mathcal{O}(x_1,\ldots,x_n)\rangle = -\frac{e^2}{16\pi^2}\langle\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)\mathcal{O}(x_1,\ldots,x_n)\rangle. \tag{30.70}$$

We often abbreviate this with

$$\partial_\mu J_\mu^5 = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}, \tag{30.71}$$

which agrees with Eq. (30.22). This equation confirms the interpretation of the chiral anomaly as due to non-invariance of the path integral measure.

Since this derivation did not appear to use perturbation theory, it seems to imply that the anomaly equation, Eq. (30.71), is exact. Indeed, the conclusion is correct:

> The chiral anomaly is 1-loop exact.

But the logic is flawed. In fact, the path integral transformation amounts to a 1-loop computation, as can be seen from Eq. (30.67) or by restoring factors of $\hbar$ (the correspondence between functional determinants and loops will be explored in Chapters 33 and 34). Thus, a more accurate statement is *because* the anomaly is exact at 1-loop, the measure transformation gives the correct answer. The 1-loop exactness of the chiral anomaly was first proposed by Adler and Bell using diagrammatic arguments. Its most satisfying proof uses topological arguments (see for example [Nakahara, 2003] or [Weinberg, 1996] for details).

30.4 Gauge anomalies in the Standard Model

In this section, we will check that the currents associated with the $\mathrm{SU}(3)_{\mathrm{QCD}} \times \mathrm{SU}(2)_{\mathrm{weak}} \times \mathrm{U}(1)_Y$ gauge symmetries of the Standard Model are non-anomalous. If we write these three currents as J_μ^{QCD}, J_μ^{weak} and J_μ^Y, then we have to show that $\partial_\mu \langle J_\mu^j J_\alpha^k J_\nu^l \rangle = 0$ for j, k, l any of the forces. This is easiest to do by reading charges or anomaly coefficients from the triangle diagrams.

When all three currents involved are associated with $\mathrm{U}(1)_Y$, we call the putative anomaly the $\mathrm{U}(1)_Y^3$ anomaly. It is easy to check that this vanishes. As we saw in Eq. (30.53), left-handed Weyl fermions and right-handed Weyl fermions contribute to the anomaly with opposite signs. Therefore, we have

$$\partial_\mu J_Y^\mu = \left(\sum_{\text{left}} Y_l^3 - \sum_{\text{right}} Y_r^3 \right) \frac{g'^2}{32\pi^2} \varepsilon^{\mu\nu\alpha\beta} B_{\mu\nu} B_{\alpha\beta}, \tag{30.72}$$

where $B_{\mu\nu}$ is the field strength for $\mathrm{U}(1)_Y$. The vanishing of the $\mathrm{U}(1)_Y^3$ anomaly requires

$$0 = \left(2Y_L^3 - Y_e^3 - Y_\nu^3\right) + 3\left(2Y_Q^3 - Y_u^3 - Y_d^3\right). \tag{30.73}$$

Here, $Y_L, Y_e, Y_\nu, Y_Q, Y_u$ and Y_d are the hypercharges for the left-handed leptons, the right-handed electrons (or muon or tauon), the right-handed neutrinos (assuming they exist), the left-handed quarks, the right-handed up-type quarks and the right-handed down-type quarks, respectively. As derived in Chapter 29, these charges are (see Table 29.1)

$$Y_L = -\frac{1}{2}, \quad Y_e = -1, \quad Y_\nu = 0, \quad Y_Q = \frac{1}{6}, \quad Y_u = \frac{2}{3}, \quad Y_d = -\frac{1}{3}. \tag{30.74}$$

Plugging in to Eq. (30.73), we find that the anomaly in fact vanishes. Note that the anomaly would vanish for any number of generations, but that it does not vanish for the quarks or leptons alone.

By the way, one can also trivially check that the $\mathrm{U}(1)_{\mathrm{EM}}^3$ anomaly vanishes in QED. In QED, all the left- and right-handed charged particles are Dirac, and hence have the same charges (QED is non-chiral). Thus, in QED, $\sum_{\text{left}} Q_L^3 = \sum_{\text{right}} Q_R^3$. That the $\mathrm{U}(1)_{\mathrm{EM}}^3$ anomaly vanishes also follows from the vanishing of anomalies in the electroweak theory, which we have nearly shown.