# 33 Effective actions and Schwinger proper time

We have mentioned effective actions a few times already. For example, the effective action for the 4-Fermi theory is derived from the Standard Model by *integrating out* the $W$ and $Z$ bosons. It is an *effective* action since it is valid only in some regime, in this case for energies less than $m_W$. More generally, an effective action is one that gives the same results as a given action but has different degrees of freedom. For the 4-Fermi theory, the effective action does not have the $W$ and $Z$ bosons. In this chapter we will develop powerful tools to calculate effective actions more generally. We will discuss three ways to calculate effective actions: through matching (or the operator product expansion), through field-dependent expectation values using Schwinger proper time, and with functional determinants coming from Feynman path integrals.

The first step is to define what we mean by an effective action. The term **effective action**, denoted by $\Gamma$, generally refers to a functional of fields (like any action) defined to give the same Green's functions and $S$-matrix elements as a given action $S$, which is often called the action for the **full theory**. We write $\Gamma = \int d^4x\, \mathcal{L}_{\text{eff}}(x)$, where $\mathcal{L}_{\text{eff}}$ is called the **effective Lagrangian**. Differences between $\Gamma$ and $S$ include that $\Gamma$ often has fewer fields, is non-renormalizable, and only has a limited range of validity. When a field is in the full theory but not in the effective action, we say it has been **integrated out**.

The advantage of using effective actions over full theory actions is that by focusing only on the relevant degrees of freedom for a given problem calculations are often easier. For example, in Section 31.3 we saw that in the 4-Fermi theory large logarithmic corrections to $b \to c\bar{d}u$ decays of the form $\alpha_s^n \ln^n \frac{m_W}{m_b}$ could be summed to all orders in perturbation theory. The analogous calculation in the full Standard Model would have been a nightmare.

The effective action we will focus on for the majority of this chapter is the one arising from integrating out a fermion of mass $m$ in QED. We can define this effective action $\Gamma[A_\mu]$ by

$$\int \mathcal{D}A \exp(i\Gamma[A_\mu]) \equiv \int \mathcal{D}A\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \exp\left[i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(i\not{D} - m)\psi\right)\right]. \tag{33.1}$$

When $A_\mu$ corresponds to a constant electromagnetic field, $\mathcal{L}_{\text{eff}}[A]$ is called the Euler–Heisenberg Lagrangian. The Euler–Heisenberg Lagrangian is amazing: it gives us the QED $\beta$-function, Schwinger pair creation, scalar and pseudoscalar decay rates, the chiral anomaly, and the low-energy limit for scattering $n$ photons, including the light-by-light scattering cross section. As we will see, the Euler–Heisenberg Lagrangian can be calculated to all orders in $\alpha_e$ using techniques from non-relativistic quantum mechanics.

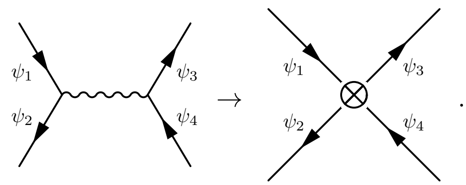
## 33.1 Effective actions from matching

So far, we have only discussed how effective actions can be calculated through matching. This approach requires that matrix elements of states agree in the full and effective theories. For example, in the 4-Fermi theory, we asked that

$$\langle\Omega|T\{\bar{\psi}\psi\bar{\psi}\psi\}|\Omega\rangle_S = \langle\Omega|T\{\bar{\psi}\psi\bar{\psi}\psi\}|\Omega\rangle_\Gamma, \tag{33.2}$$

where the subscript on the correlation function indicates the action used to calculate it. Writing the effective Lagrangian as a sum over operators $\mathcal{L}_{\text{eff}}(x) = \sum C_i \mathcal{O}_i(x)$ we were able to determine the Wilson coefficients $C_i$ by asking that Eq. (33.2) hold order-by-order in perturbation theory. One-loop matching in the 4-Fermi theory was discussed in Section 31.3. Other examples of matching that we considered include the Chiral Lagrangian (Section 28.2.2) and deep inelastic scattering (Section 32.4).

In the 4-Fermi theory and for deep inelastic scattering, we matched by expanding propagators $\frac{1}{p^2-m_W^2}$ or $\frac{1}{p^2+Q^2}$ respectively (see Eqs. (32.70) and (32.71)). The reason one can expand propagators to derive an effective Lagrangian is because when a scale such as $m_W$ or $Q$ is taken large, the propagator can only propagate over a small distance. In terms of Feynman diagrams, we expand an exchange graph in a set of local interactions:




(33.3)

To see how this works in position space, consider matching a Yukawa theory with a massive scalar,

$$\mathcal{L}_Y = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}\phi(\Box + m^2)\phi + \lambda\phi\bar{\psi}\psi, \tag{33.4}$$

to an effective Lagrangian $\mathcal{L}_{\text{eff}}$ which lacks that scalar and is useful for energies much less than $m$. For large $m$, fluctuations of $\phi$ around its classical configuration are highly suppressed. Thus, to leading order we can assume $\phi$ satisfies its classical equations of motion, $\phi = \frac{\lambda}{\Box+m^2}\bar{\psi}\psi$, and that loops of $\phi$ are small corrections. Plugging the classical solution back into the Lagrangian gives

$$\mathcal{L}_{\text{eff}} = i\bar{\psi}\partial\!\!\!/\psi + \frac{\lambda^2}{2}\bar{\psi}\psi\frac{1}{\Box + m^2}\bar{\psi}\psi. \tag{33.5}$$

In this way $\mathcal{L}_{\text{eff}}$ is guaranteed to give the same correlation functions as $\mathcal{L}_Y$ but has no $\phi$ field in it. As long as $m$ is larger than typical momentum scales, we can also Taylor expand this non-local effective Lagrangian in a series of local operators:

$$\mathcal{L}_{\text{eff}} = i\bar{\psi}\partial\!\!\!/\psi + \frac{\lambda^2}{2m^2}\bar{\psi}\psi\bar{\psi}\psi - \frac{\lambda^2}{2m^4}\bar{\psi}\psi\Box\bar{\psi}\psi + \cdots. \tag{33.6}$$

If $\phi$ were the $W$ and $Z$, this would give the 4-Fermi theory supplemented by additional operators that have effects suppressed by powers of $\frac{E^2}{m_W^2}$ at low energy.

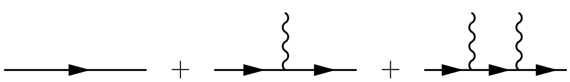
Setting $\phi$ to its classical equations of motion amounts to taking the steepest descent approximation in the path integral. To integrate out $\phi$ to all orders, we have to perform the path integral exactly. Thus, we can define the effective action as

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\exp\left(i\int d^4x\,\mathcal{L}_{\text{eff}}[\psi,\bar{\psi}]\right) = \int \mathcal{D}\phi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\exp\left(i\int d^4x\,\mathcal{L}_Y[\phi,\psi,\bar{\psi}]\right), \tag{33.7}$$

which connects back to the definition given in Eq. (33.1).

## 33.2 Effective actions from Schwinger proper time

The next method we discuss for computing effective actions is through Schwinger proper time. The idea here is to evaluate the propagator for the particle we want to integrate out as a functional of the other fields. Pictorially, we can write this as

$G_A(x,y) =$  $+\cdots$. (33.8)

Then, when we integrate out the field, we will generate an infinite set of interactions among the other fields.

The key to Schwinger's proper-time formalism is the mathematical identity

$$\frac{i}{A+i\varepsilon} = \int_0^\infty ds\, e^{is(A+i\varepsilon)}, \tag{33.9}$$

which holds for $A \in \mathbb{R}$ and $\varepsilon > 0$ (see Appendix B). This lets us write the Feynman propagator for a scalar as

$$\begin{aligned} D_F(x,y) &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \int_0^\infty ds\, e^{is\,(p^2 - m^2 + i\varepsilon)}. \end{aligned} \tag{33.10}$$

The integral over $d^4p$ is Gaussian and can be done exactly using Eq. (14.7) with $A = -2isg^{\mu\nu}$, giving

$$D_F(x,y) = \frac{-i}{16\pi^2}\int_0^\infty \frac{ds}{s^2} e^{-i\left[\frac{(x-y)^2}{4s} + sm^2 - i\varepsilon s\right]}, \tag{33.11}$$

which is an occasionally useful representation of the propagator. For $m = 0$ it provides a shortcut to the position-space Feynman propagator $D_F(x,y) = -\frac{1}{4\pi^2}\frac{1}{(x-y)^2 - i\varepsilon}$.

An alternative to performing the integral over $p$ directly is first to introduce a one-particle Hilbert space spanned by $|x\rangle$, as in non-relativistic quantum mechanics. This lets us write $\langle p|x\rangle = e^{ipx}$. Then, from Eq. (33.10) we get

$$D_F(x,y) = \int \frac{d^4p}{(2\pi)^4} \langle y|p\rangle \int_0^\infty ds\, e^{is\left(p^2-m^2+i\varepsilon\right)} \langle p|x\rangle. \tag{33.12}$$

The analogy with quantum mechanics can be taken even further. Introduce momentum operators $\hat{p}^\mu$ with $\hat{p}^\mu|p\rangle = p^\mu|p\rangle$ and define $\hat{H} = -\hat{p}^2$. Then $e^{isp^2}\langle p|x\rangle = \langle p|e^{-is\hat{H}}|x\rangle$. This lets us use $(2\pi)^{-4}\int d^4p\,|p\rangle\langle p| = \mathbb{1}$ in Eq. (33.12) to get

$$D_F(x,y) = \int_0^\infty ds\, e^{-s\varepsilon} e^{-ism^2} \langle y|e^{-is\hat{H}}|x\rangle \equiv \int_0^\infty ds\, e^{-s\varepsilon} e^{-ism^2} \langle y;0|x;s\rangle, \tag{33.13}$$

where $|x;s\rangle \equiv e^{-is\hat{H}}|x\rangle$. In the second step, we have interpreted $\hat{H}$ as a Hamiltonian and $s$ as a time variable known as **Schwinger proper time**.[1] Schwinger proper time gives an intuitive interpretation of a propagator:

> A propagator is the amplitude for a particle to propagate from $x$ to $y$ in proper time $s$, integrated over $s$.

One has to be careful interpreting $\hat{H}$ however, since it conventionally includes only the $p$ dependence and not the $m$ dependence (as $\hat{H} = m^2 - \hat{p}^2$ would).

We can go even further into quantum mechanics by defining the Green's function as an operator matrix element. Define the Green's function operator for a massive scalar as

$$\hat{G} \equiv \frac{i}{\hat{p}^2 - m^2 + i\varepsilon}. \tag{33.14}$$

Then the Feynman propagator is

$$D_F(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon} = \int \frac{d^4p}{(2\pi)^4} \langle y|p\rangle\langle p| \frac{i}{\hat{p}^2 - m^2 + i\varepsilon}|x\rangle$$
$$= \langle y|\hat{G}|x\rangle. \tag{33.15}$$

Or we can go directly to proper time, without ever introducing the $p$ integral, through Eq. (33.9):

$$D_F(x,y) = \langle y|\hat{G}|x\rangle = \int_0^\infty ds\, e^{-s\varepsilon} e^{-ism^2} \langle y|e^{-i\hat{H}s}|x\rangle, \tag{33.16}$$

where $\hat{H} = -\hat{p}^2$ as before.

By the way, when you have two propagators, as in a loop, the relevant identity is

$$\frac{1}{AB} = -\int_0^\infty ds \int_0^\infty dt\, e^{isA+itB} \tag{33.17}$$

(the $i\varepsilon$ factors are implicit). If we then write $s = x\tau$ and $t = (1-x)\tau$, so that $s$ and $t$ are the fractions $x$ and $(1-x)$ of the total proper time $\tau$, this becomes

$$\frac{1}{AB} = -\int_0^1 dx \int_0^\infty \tau\, d\tau\, e^{i\tau(xA+(1-x)B)} = \int_0^1 dx \frac{1}{[Ax + B\,(1-x)]^2}, \tag{33.18}$$

[1] To understand why $s$ is called a proper time, recall from relativity that proper time $s$ is defined by the differential $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Since $\hat{H} = -g_{\mu\nu}\hat{p}^\mu\hat{p}^\nu$, it naturally generates translations in proper time through $g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$.

which is a Feynman parameter integral. Thus, in a loop, each particle has its own proper time, $s$ or $t$, which denote how long each particle has taken to get around its part of the loop. Then the Feynman parameter $x = \frac{s}{s+t}$ is how far one particle is behind the other one.

### 33.2.1 Background fields

Now suppose a field $\phi$ interacts with a photon field, through the usual scalar QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \phi^\star\left(D^2 + m^2\right)\phi, \tag{33.19}$$

with $D_\mu = \partial_\mu + ieA_\mu$. As a step towards calculating the Euler–Heisenberg Lagrangian, we will need the scalar propagator in the presence of a fixed external $A_\mu$ field. We write $\langle A|\cdots|A\rangle$ instead of $\langle\Omega|\cdots|\Omega\rangle$ when matrix elements are taken in the presence of an external field rather than the vacuum. Thus, the propagator in the presence of an external field $A_\mu$ is written as

$$G_A(x,y) = \langle A|T\{\phi(y)\phi^\star(x)\}|A\rangle. \tag{33.20}$$

Using operator notation, we use $\partial_\mu \to -i\hat{p}_\mu$ to define

$$\hat{G}_A = \frac{i}{(\hat{p} - eA(\hat{x}))^2 - m^2 + i\varepsilon}. \tag{33.21}$$

This equation illustrates an advantage of the quantum mechanics operator formalism over Feynman diagrams: we can work in position and momentum space at the same time, through operators such as $\hat{p} - eA(\hat{x})$.

Then, as in Eq. (33.15), we have

$$G_A(x,y) = \langle y|\hat{G}_A|x\rangle = \langle y|\,\frac{i}{(\hat{p} - eA(\hat{x}))^2 - m^2 + i\varepsilon}|x\rangle = \int ds\, e^{-s\varepsilon} e^{-ism^2}\langle y|e^{-i\hat{H}s}|x\rangle, \tag{33.22}$$

where now

$$\hat{H} = -(\hat{p} - eA(\hat{x}))^2. \tag{33.23}$$

So we get the same formula as for the free theory, but with a different Hamiltonian. The interpretation of Eq. (33.22) is that $G_A(x,y)$ describes the evolution of $\phi$ from $x$ to $y$ in time $s$, including all possible interactions with a field $A_\mu$ over all possible times $s$. This is shown diagrammatically in Eq. (33.8).

For a spinor, we want to evaluate

$$G_A(x,y) = \langle A|T\{\psi(y)\bar{\psi}(x)\}|A\rangle. \tag{33.24}$$

First, recall from Eq. (10.106) that

$$\not{D}^2 = D_\mu^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}. \tag{33.25}$$

We used this identity in Chapter 10 to show that Dirac spinors satisfy the Klein–Gordon equation with an additional magnetic moment term. Here, the $F_{\mu\nu}\sigma^{\mu\nu}$ term will again produce the differences between the scalar and Dirac spinor cases of quantities we calculated. Then, in momentum space, we have

$$(\hat{\not{p}} - e\not{A}(\hat{x}))^2 = (\hat{p} - eA(\hat{x}))^2 - \frac{e}{2}F_{\mu\nu}(\hat{x})\sigma^{\mu\nu}. \tag{33.26}$$

This identity lets us write the spinor Green's function operator as

$$\begin{aligned}\hat{G}_A &= \frac{i}{\hat{\not{p}} - e\not{A}(\hat{x}) - m + i\varepsilon} \\ &= \left(\hat{\not{p}} - e\not{A}(\hat{x}) + m\right)\frac{i}{(\hat{p} - eA(\hat{x}))^2 - \frac{e}{2}F_{\mu\nu}(\hat{x})\sigma^{\mu\nu} - m^2 + i\varepsilon},\end{aligned} \tag{33.27}$$

and so the Dirac propagator is

$$G_A(x,y) = \langle y|\frac{i}{\hat{\not{p}} - e\not{A} - m + i\varepsilon}|x\rangle = \int_0^\infty ds\, e^{-s\varepsilon}e^{-ism^2}\langle y|(\hat{\not{p}} - e\not{A}(\hat{x}) + m)e^{-i\hat{H}s}|x\rangle \tag{33.28}$$

as before, but now with

$$\hat{H} = -(\hat{p}^\mu - eA^\mu(\hat{x}))^2 + \frac{e}{2}F_{\mu\nu}(\hat{x})\sigma^{\mu\nu}. \tag{33.29}$$

Note that there is no Dirac trace here, since the Green's function is a matrix in spinor space.

### 33.2.2 Field-dependent expectation values

To connect to effective actions, recall from Section 33.1 that to integrate out a field at tree-level we set it equal to its equations of motion. Another way to phrase this procedure is that we set the field equal to a configuration for which the Lagrangian has a minimum. Now, classically, we can always expect to find the field at the minimum. So the minimum can be thought of as a classical expectation. The generalization to the quantum theory is to replace a field by its quantum vacuum expectation value:

$$\phi \to \langle\Omega|\phi|\Omega\rangle\,. \tag{33.30}$$

The classical and quantum expectation values agree at tree-level, but can be different when loops or non-perturbative effects are included. We will consider how the vacuum can be destabilized by quantum effects in Chapter 34. Our focus here is not on the expectation value in the vacuum, but in the presence of a fixed electromagnetic field. Thus, in a background field, we can integrate out $\phi$ by replacing $\phi \to \langle A|\phi|A\rangle$.

Let us go straight to the fermion case. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(i\not{\partial} - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi. \tag{33.31}$$

We now want to replace this by the effective Lagrangian where the current that $A_\mu$ couples to is replaced by its expectation value in the given fixed configuration, which we are denoting as $A_\mu$:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}^2 - eA_\mu J_A^\mu, \tag{33.32}$$

where

$$J^\mu_A \equiv \langle A|\bar\psi(x)\gamma^\mu\psi(x)|A\rangle. \tag{33.33}$$

This is not a vacuum matrix element, but a matrix element in the presence of a given state $|A\rangle$.

Now we can calculate $J^\mu_A$ using Schwinger proper time. First note that $A = 0$ is the vacuum, so $J^\mu_0$ should reduce to the propagator $G(x, y)$ with $x = y$ when the field is turned off. Indeed, being explicit about the spin indices

$$J^\mu_0(x) = \langle\Omega|\bar\psi_{\dot\alpha}(x)\gamma^\mu_{\dot\alpha\alpha}\psi_\alpha(x)|\Omega\rangle = -\mathrm{Tr}\Big[\langle\Omega|\psi_\alpha(x)\bar\psi_{\dot\alpha}(x)\gamma^\mu_{\dot\alpha\beta}|\Omega\rangle\Big] \equiv -\mathrm{Tr}\,\langle x|\hat G\gamma^\mu|x\rangle. \tag{33.34}$$

The third form is meant to indicate that the trace of the matrix $\left[\psi\bar\psi\gamma^\mu\right]_{\alpha\beta}$ is being taken. In the presence of a non-zero $A$ field, we just have to replace this by the propagator in the $A_\mu$ background:

$$J^\mu_A(x) = -\mathrm{Tr}\,\langle x|\hat G_A\gamma^\mu|x\rangle, \tag{33.35}$$

where $\hat G_A$ is the Green's function in Eq. (33.27). So,

$$\begin{aligned} J^\mu_A &= -\mathrm{Tr}\left[\int_0^\infty ds\, e^{-s\varepsilon}e^{-ism^2}\langle x|\gamma^\mu(\not p - e\not A + m)e^{-i\hat H s}|x\rangle\right] \\ &= -\int_0^\infty ds\, e^{-s\varepsilon}e^{-ism^2}\langle x|\mathrm{Tr}\left[\gamma^\mu(\not p - e\not A)e^{i((\hat p - eA)^2 - \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu})s}\right]|x\rangle, \end{aligned} \tag{33.36}$$

where we have used that Tr of an odd number of $\gamma$-matrices is zero. Next, note that the current is itself a variation:

$$J^\mu_A = -\frac{i}{2e}\frac{\partial}{\partial A_\mu}\int_0^\infty \frac{ds}{s}e^{-s\varepsilon}e^{-ism^2}\mathrm{Tr}\Big[\langle x|e^{-i\hat H s}|x\rangle\Big]. \tag{33.37}$$

Integrating both sides with respect to $A_\mu$ and using Eq. (33.32) gives

$$\mathcal{L}_{\mathrm{eff}}(x) = -\frac{1}{4}F^2_{\mu\nu}(x) + \frac{i}{2}\int_0^\infty \frac{ds}{s}e^{-s\varepsilon}e^{-ism^2}\mathrm{Tr}\Big[\langle x|e^{-i\hat H s}|x\rangle\Big], \tag{33.38}$$

which is only a function of the background field $A_\mu$. For a spinor, $\hat H$ is given in Eq. (33.29).

For a complex scalar, the effective Lagrangian has a similar form:

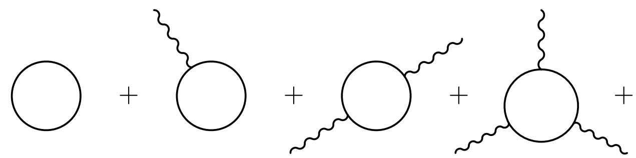
$$\mathcal{L}_{\mathrm{eff}}(x) = -\frac{1}{4}F^2_{\mu\nu}(x) - i\int_0^\infty \frac{ds}{s}e^{-s\varepsilon}e^{-ism^2}\langle x|e^{-i\hat H s}|x\rangle, \tag{33.39}$$

with $\hat H = -(\hat p - eA(\hat x))^2$ as in Eq. (33.23). The scalar case is actually more difficult to derive than the spinor case using Schwinger's method because of the $A^2_\mu\phi^\star\phi$ term in the scalar QED Lagrangian. We produce this Lagrangian using Feynman path integrals in Eq. (33.52) below.

### 33.2.3 Interpretation and cross check

Up to an extra factor of $\frac{1}{s}$, the proper-time integral in Eq. (33.39) looks just like $\langle x|\hat{G}_A|y\rangle$ in Eq. (33.22) with $x = y$. This is easy to understand: the effective action sums closed loops, where the particle propagates back to where it started after some proper time $s$. That is, it is an integral over $\langle x;0|x;s\rangle$. In terms of Feynman diagrams, the effective action includes all diagrams with any number of external photons and one closed fermion loop:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}^2 \ + $$



$$+\cdots . \tag{33.40}$$

The physical interpretation of the expectation value $\langle x|e^{-i\hat{H}s}|x\rangle = \langle x;0|x;s\rangle$ in Eq. (33.38) is therefore that it is the amplitude for a particle to go around a loop in proper time $s$ based on evolution with the Hamiltonian $\hat{H}$.

Note that the first diagram in Eq. (33.40) does not involve any photons at all, thus it should represent the vacuum energy of the system. This provides a nice consistency check. Setting $A = 0$, to get just the first diagram, the effective action becomes (in the complex scalar case)

$$\Gamma[0] = -i\int d^4x \int_0^\infty \frac{ds}{s} e^{-s\varepsilon} e^{-ism^2} \langle x|e^{i\hat{p}^2 s}|x\rangle. \tag{33.41}$$

Inserting $\mathbb{1} = \int \frac{d^4k}{(2\pi)^4} |k\rangle\langle k|$ we find

$$\Gamma[0] = -iVT \int_0^\infty \frac{ds}{s} \int \frac{d^4k}{(2\pi)^4} \exp\Big[i(k_0^2 - \vec{k}^2 - m^2 + i\varepsilon)s\Big], \tag{33.42}$$

where $VT$ is the volume of space-time. It is convenient to remove this factor by writing $\Gamma[0] = -(VT)V_{\text{eff}}$ with $V_{\text{eff}}$ an effective potential energy density, which in this case is just a constant.

The integral over proper time is divergent from the $s \sim 0$ region, corresponding to where the loop has zero proper length. However, Schwinger proper time conveniently gives us a Lorentz-invariant and gauge-invariant way to regulate such divergences: cut off the integral for $s > s_0$. To evaluate $V_{\text{eff}}$, we Wick rotate $k_0 \to ik_0$ and can integrate over the imaginary axis. This gives

$$\begin{aligned} V_{\text{eff}} &= -\int_{s_0}^\infty \frac{ds}{s} \int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \exp\Big[-i(k_0^2 + \vec{k}^2 + m^2)s\Big] \\ &= -\frac{1}{2\sqrt{\pi}} \int \frac{d^3k}{(2\pi)^3} \int_{s_0}^\infty \frac{ds}{s^{3/2}} \exp\Big[-(\vec{k}^2 + m^2)s\Big], \end{aligned} \tag{33.43}$$

where we have replaced $s \to -is$ in the second step. Then we find

$$V_{\text{eff}} = \int \frac{d^3k}{(2\pi)^3} \left( -\frac{1}{\sqrt{\pi s_0}} + \sqrt{\vec{k}^2 + m^2} + \mathcal{O}(\sqrt{s_0}) \right). \tag{33.44}$$

The $-\frac{1}{\sqrt{\pi s_0}}$ is a divergent constant, corresponding to an extrinsic cutoff-dependent vacuum energy. This can be removed with a vacuum energy counterterm. The important term is in the integral over $\sqrt{\vec{k}^2+m^2}=\omega_k$, which counts the ground-state energies of the modes. It was this sum, not the constant, that led to the Casimir force discussed in Chapter 15.

Note that we get $\omega_k$ instead of $\frac{1}{2}\omega_k$ since this is the effective action for a complex scalar that has twice the energy of a real scalar. For a Dirac fermion, the calculation is identical, since $\hat{H}=-\hat{p}^2$ in both cases when $A=0$. The only difference is that the Dirac trace and $-\frac{1}{2}$ in Eq. (33.38) give a factor of $4(-\frac{1}{2})=-2$ compared to the scalar case in Eq. (33.39). The minus sign is consistent with a fermion loop and the factor of 2 is consistent with a Dirac spinor having twice the number of degrees of freedom of a complex scalar. These are the same results we found in Section 12.5 by computing the energy density from the energy-momentum tensor. One consequence is that in a theory with a Weyl fermion and a complex scalar of the same mass, such as in theories with supersymmetry, the vacuum energy is zero.

## 33.3 Effective actions from Feynman path integrals

An alternative approach to calculating the effective action is based on the Feynman path integral. Here we want to integrate over some fields by performing the path integral. For scalar QED, integrating out the scalar means

$$\int \mathcal{D}A\exp(i\Gamma[A])=\int \mathcal{D}A\,\mathcal{D}\phi\,\mathcal{D}\phi^\star\exp\left[i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^2-\phi^\star(D^2+m^2)\phi\right)\right]. \tag{33.45}$$

In this case, since the original action is quadratic in $\phi$, we can evaluate the path integral exactly. We will ignore the $i\varepsilon$ in this section for simplicity.

Recall the general formula from Problem 14.1:

$$\int \mathcal{D}\phi^\star\mathcal{D}\phi\exp\left[i\int d^4x(\phi^\star M\phi+JM)\right]=\mathcal{N}\frac{1}{\det M}\exp(iJM^{-1}J), \tag{33.46}$$

where $\mathcal{N}$ is some (infinite) normalization constant. Thus, for the scalar QED Lagrangian we find

$$\int \mathcal{D}A\exp(i\Gamma[A])=\mathcal{N}\int\mathcal{D}A\exp\left[i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^2\right)\right]\frac{1}{\det(-D^2-m^2)}. \tag{33.47}$$

This equation will be satisfied if

$$\exp\left[i\Gamma[A]+i\int d^4x\frac{1}{4}F_{\mu\nu}^2\right]=\mathcal{N}\frac{1}{\det(-D^2-m^2)}. \tag{33.48}$$

To make this notation somewhat less opaque, we can turn this mysterious determinant into a sum by noting that

$$i\Gamma[A] + i\int d^4x \frac{1}{4}F_{\mu\nu}^2 - \ln\mathcal{N} = -\ln\big[\det(-D^2 - m^2)\big] = -\mathrm{tr}\big[\ln(-D^2 - m^2)\big]. \tag{33.49}$$

The trace is a sum over eigenvalues, in this case, eigenvalues of $-\ln\big(-D^2 - m^2\big)$. One can either evaluate this trace in momentum space, as will be discussed in Chapter 34, or in position space, as we discuss here. The beautiful thing about a trace is that it is basis independent. So we can just evaluate the sum on position eigenstates. That is, using the quantum mechanics notation from Section 33.2 we have

$$i\Gamma[A] = \int d^4x \left[-\frac{i}{4}F_{\mu\nu}^2 - \langle x|\ln(-D^2 - m^2)|x\rangle\right] + \ln\mathcal{N}. \tag{33.50}$$

To connect to Schwinger proper time, take a derivative with respect to $m^2$ and introduce a Schwinger parameter. Then,

$$\frac{d}{dm^2}\langle x|\ln(-D^2 - m^2)|x\rangle = -\langle x|\frac{1}{-D^2 - m^2}|x\rangle = i\int_0^\infty ds\, e^{-ism^2}\langle x|e^{-i\hat{H}s}|x\rangle, \tag{33.51}$$

with $\hat{H} = -\left(\hat{p} - eA(\hat{x})\right)^2$ as in Eq. (33.23). Integrating over $m^2$ and restoring the $i\varepsilon$, which we have been ignoring in this section, gives

$$\mathcal{L}_{\text{eff}}(x) = -\frac{1}{4}F_{\mu\nu}^2 - i\int_0^\infty \frac{ds}{s} e^{-s\varepsilon} e^{-ism^2}\langle x|e^{-i\hat{H}s}|x\rangle + \text{ const}, \tag{33.52}$$

where the integration constant and $\ln\mathcal{N}$ have been combined. Physics is unaffected by these constants, and indeed we will exploit the fact that $\mathcal{L}_{\text{eff}}$ can be shifted by a constant to remove infinities when $\mathcal{L}_{\text{eff}}$ is renormalized.

## 33.3.1 Fermions

For fermions, we need to evaluate

$$\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp\left(i\int d^4x\,\bar{\psi}(i\not{D} - m)\psi\right) = \mathcal{N}\det(i\not{D} - m). \tag{33.53}$$

Thus,

$$i\Gamma[A] = i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^2\right) + \mathrm{Tr}\big[\mathrm{tr}(\ln(i\not{D} - m))\big] + \text{ const}, \tag{33.54}$$

where Tr indicates a Dirac trace and tr is the normal integral over $x^\mu$ or $p^\mu$. The effective Lagrangian is then

$$\mathcal{L}_{\text{eff}}(x) = -\frac{1}{4}F_{\mu\nu}^2 - i\mathrm{Tr}\big[\langle x|\ln\big(i\not{D} - m\big)|x\rangle\big] + \text{ const}. \tag{33.55}$$

As before, we take a derivative with respect to $m^2$:

$$\frac{d}{dm^2}\mathcal{L}_{\text{eff}}(x) = \frac{i}{2m}\text{Tr}\langle x|\frac{i\not{D}+m}{-\not{D}^2-m^2}|x\rangle = \frac{i}{2}\text{Tr}\left[\langle x|\frac{1}{-\not{D}^2-m^2}|x\rangle\right]$$
$$= \frac{1}{2}\int_0^\infty ds\, e^{-ism^2}\text{Tr}\left[\langle x|e^{-i\not{D}^2 s}|x\rangle\right], \tag{33.56}$$

where we have used in the second step that the trace of an odd number of $\gamma$-matrices is 0. Integrating over $m^2$ gives

$$\mathcal{L}_{\text{eff}}(x) = -\frac{1}{4}F_{\mu\nu}^2 + \frac{i}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2}\text{Tr}\left[\langle x|e^{-i\not{D}^2 s}|x\rangle\right] + \text{ const.} \tag{33.57}$$

Using Eq. (33.25), we then get

$$\mathcal{L}_{\text{eff}}(x) = -\frac{1}{4}F_{\mu\nu}^2 + \frac{i}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2}\text{Tr}\left[\langle x|e^{i\left[(\hat{p}-eA(\hat{x}))^2-\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}\right]s}|x\rangle\right] + \text{ const,} \tag{33.58}$$

which agrees with Eq. (33.38).

Another way to obtain this result is to observe that

$$\text{Tr}\langle x|\ln\big(i\not{D}-m\big)|x\rangle = \text{Tr}\langle x|\ln(-i\not{D}-m)|x\rangle\,. \tag{33.59}$$

So averaging the two gives

$$\text{Tr}\langle x|\ln(i\not{D}-m)|x\rangle = \frac{1}{2}\text{Tr}\langle x|\ln\big(-\not{D}^2-m^2\big)|x\rangle. \tag{33.60}$$

We can write this in terms of Schwinger parameters using the identity

$$\int_{s_0}^\infty \frac{ds}{s}e^{isA} = -\ln(A) - \ln s_0 + \text{ finite,} \tag{33.61}$$

which holds as $s_0 \to 0$. This lets us write Eq. (33.54) with Eq. (33.60) as Eq. (33.58).

## 33.4 Euler–Heisenberg Lagrangian

Now we are ready to do some physics! We will calculate the effective action for the case of a constant background electromagnetic field $F_{\mu\nu}$ (which is not the same as constant $A_\mu$). From Eq. (33.38) we need to evaluate $\langle x|e^{-i\hat{H}s}|x\rangle$, where $\hat{H} = -(\hat{p}-eA(\hat{x}))^2+\frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}$ in the spinor case and $\hat{H} = -\left(\hat{p}-eA(\hat{x})\right)^2$ for scalars. There are a number of ways to evaluate this trace. The quickest way is to work in basis $|\psi_n\rangle$ of eigenstates of $\hat{H}$. Then we can use

$$\int d^4x\langle x|e^{-i\hat{H}s}|x\rangle = \int d^4x\sum_n\langle x|\psi_n\rangle\langle\psi_n|e^{-i\hat{H}s}|x\rangle$$
$$= \int d^4x\sum_n|\psi_n(x)|^2\, e^{-iE_n s}. \tag{33.62}$$

Thus, we just have to sum $e^{-iE_n s}$ over all the eigenvalues $E_n$ of $\hat{H}$. In this way, we reduce the problem to non-relativistic quantum mechanics. An alternative, somewhat more general, approach is discussed in Appendix 33.A.

We are interested in constant $F_{\mu\nu}$. For a constant magnetic field in the $\hat{z}$ direction, we can take $A_y = B\hat{x}$ and so the Hamiltonian becomes

$$\hat{H} = \left[-\hat{p}_t^2 + \hat{p}_x^2 + \hat{p}_z^2 + (\hat{p}_y - eB\hat{x})^2\right] \times \mathbb{1}_{4\times 4} - eB \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \tag{33.63}$$

with the $eB\sigma_z$ term being the spin–magnetic moment interaction coming from $\sigma_{\mu\nu}F^{\mu\nu}$. $\hat{H}$ has eigenstates for any values of $p_t, p_y$ and $p_z$. Writing

$$\psi_n^{p_t,p_y,p_z} = \chi_n\left(x - \frac{p_y}{eB}\right)e^{ip_t t - ip_y y - ip_z z} \tag{33.64}$$

reduces the problem to finding the eigenstates of $\hat{p}_x^2 + (eB\hat{x})^2$, which is just the non-relativistic harmonic oscillator Hamiltonian. The result is that $\chi_n$ are the harmonic oscillator wavefunctions and $n$ takes discrete values, corresponding to the **Landau levels** of a non-relativistic electron in a magnetic field. The energies are therefore

$$E_n^{p_t,p_y,p_z,\lambda} = -p_t^2 + p_z^2 + eB(2n+1) - 2eB\lambda, \tag{33.65}$$

where $\lambda = \pm\frac{1}{2}$ comes from spin being up or down in the $z$ direction.

From Eq. (33.62), we then get

$$\int d^4x \langle x|e^{-i\hat{H}s}|x\rangle = 2\int d^4x \frac{dp_t dp_y dp_z}{(2\pi)^3} \sum_{n=0}^{\infty} \sum_{\lambda=\pm\frac{1}{2}} \left|\chi_n(x - \frac{p_y}{eB})\right|^2 \times e^{i(p_t^2 - p_z^2)s} e^{-ies(2n+1)B} e^{2ieB\lambda s}, \tag{33.66}$$

where the 2 comes from $\hat{H}$ being block diagonal. To evaluate these sums and integrals, we put the system in a Euclidean box of size $L$. Then the $dt$, $dy$, and $dz$ integrals give a factor of $L^3$. The $dx$ integral just gives 1, since the wavefunctions are normalized. Because the wavefunctions depend on $x - \frac{p_y}{eB}$, unless $p_y < LeB$, the wavefunctions will shift out of the box; so the $p_y$ integral gives a factor of $eBL$. We then have

$$\begin{aligned}\int d^4x \langle x|e^{-i\hat{H}s}|x\rangle &= 2\sum_{\lambda=\pm\frac{1}{2}} e^{2iseB\lambda} \frac{eBL^4}{(2\pi)^3} \int_{-\infty}^{\infty} dp_z dp_t e^{i(p_t^2 - p_z^2)s} \sum_{n=0}^{\infty} e^{-ies(2n+1)B} \\ &= -2iL^4 \frac{eB}{8\pi^2} \frac{1}{s} \frac{\cos(esB)}{\sin(esB)}.\end{aligned} \tag{33.67}$$

This has no position dependence, since $B$ is constant. It corresponds to an effective Lagrangian as in Eq. (33.38) of the form

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{eB}{8\pi^2}\int_0^{\infty} \frac{ds}{s} e^{-s\varepsilon} e^{-ism^2} \frac{1}{s} \frac{\cos(esB)}{\sin(esB)}. \tag{33.68}$$

The calculation for a constant electric field is the same, but with $\vec{B} \to i\vec{E}$. The general Lorentz-invariant expression for the effective Lagrangian for any constant $F_{\mu\nu}$ can be written as

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s} e^{-s\varepsilon} e^{-ism^2} \frac{\text{Re}\cos(esX)}{\text{Im}\cos(esX)} F^{\mu\nu}\tilde{F}_{\mu\nu}, \tag{33.69}$$

where $X$ is a scalar function of the electric and magnetic fields defined by

$$X \equiv \sqrt{\frac{1}{2}F_{\mu\nu}^2 - \frac{i}{2}F^{\mu\nu}\tilde{F}_{\mu\nu}} = \sqrt{(\vec{B} + i\vec{E})^2}, \tag{33.70}$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. You are encouraged to check the constant $\vec{E}$ and general expression in Problem 33.1. Taking $s \to -is$ we find

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s} e^{is\varepsilon} e^{-sm^2} \frac{\text{Re}\cosh(esX)}{\text{Im}\cosh(esX)} F^{\mu\nu}\tilde{F}_{\mu\nu}. \tag{33.71}$$

In this form, the Lagrangian is more obviously real (except possibly near singularities as discussed in Section 33.4.3).

Finally, the Lagrangian should be renormalized. We use minimal subtraction. Expanding the integrand perturbatively in $e$, we find

$$\frac{\text{Re}\cosh(esX)}{\text{Im}\cosh(esX)} F^{\mu\nu}\tilde{F}_{\mu\nu} = -\frac{4}{e^2s^2} - \frac{2}{3}F_{\mu\nu}^2 + \frac{e^2s^2}{45}\left[(F_{\mu\nu}^2)^2 + \frac{7}{4}(F^{\mu\nu}\tilde{F}_{\mu\nu})^2\right] + \cdots. \tag{33.72}$$

The leading two terms result in a UV divergence from the small proper-time region of the $ds$ integral. These divergences can be regulated in a Lorentz-invariant and gauge-invariant way by simply cutting off $s > s_0$. The required counterterms are a constant and a renormalization of the leading $F_{\mu\nu}^2$ term. Thus, we remove the infinities with minimal subtraction, giving

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s} e^{is\varepsilon} e^{-sm^2} \left[\frac{\text{Re}\cosh(esX)}{\text{Im}\cosh(esX)} F^{\mu\nu}\tilde{F}_{\mu\nu} + \frac{4}{e^2s^2} + \frac{2}{3}F_{\mu\nu}^2\right]. \tag{33.73}$$

This is the **Euler–Heisenberg Lagrangian**. It is the renormalized effective action arising from integrating out a massive fermion for constant $F_{\mu\nu}$. It is worth emphasizing that this effective Lagrangian is non-perturbative in $e$. It encodes an infinite number of 1-loop diagrams, as in Eq. (33.40), and a tremendous amount of physics. We will go through a number of applications below.

In Appendix 33.A, we derive this Lagrangian more slowly, using Schwinger's original method. The basic idea is to calculate $\langle y|e^{-i\hat{H}s}|x\rangle = \langle y;0|x;s\rangle$ by solving the differential equation

$$i\partial_s\langle y;0|x;s\rangle = i\partial_s\,\langle y;0|e^{-i\hat{H}s}|x;0\rangle = \langle y;0|\hat{H}|x;s\rangle. \tag{33.74}$$

The Heisenberg equations of motion $\frac{d}{ds}\hat{x}^\mu = i\,[\hat{H},\hat{x}^\mu]$ and $\frac{d}{ds}\hat{p}^\mu = i\,[\hat{H},\hat{p}^\mu]$ are used to get an explicit form for $\hat{x}^\mu(s)$ and $\hat{p}^\mu(s)$ and therefore $\hat{H}(s)$. This method of calculation produces the full Green's function $G(x,y) = \langle y;0|x;s\rangle$ , which is more generally useful than the effective action alone. For $x = y$, which is relevant for the effective action, the differential equation reduces to (cf. Eq. (33.A.150)):

$$i\partial_s\langle x;0|x;s\rangle = -\mathrm{tr}\left[\frac{i}{2}e\mathbf{F}\coth(es\mathbf{F}) + \frac{e}{2}\boldsymbol{\sigma}\mathbf{F}\right]\langle x;0|x;s\rangle\,, \tag{33.75}$$

where $\mathbf{F} = F_{\mu\nu}$ and $\boldsymbol{\sigma} = \sigma_{\mu\nu}$ are matrices. The solution with appropriate boundary conditions is

$$\begin{aligned}\langle x;0|x;s\rangle &= \frac{-i}{16\pi^2}\frac{1}{s^2}\exp\left(-\frac{1}{2}\mathrm{tr}\ln\left[\frac{\sinh es\mathbf{F}}{es\mathbf{F}}\right] - i\frac{es}{2}\sigma_{\mu\nu}F^{\mu\nu}\right)\\ &= -i\frac{e^2}{64\pi^2}\frac{F^{\mu\nu}\tilde{F}_{\mu\nu}}{\mathrm{Im}\cos(esX)}\exp\left(-i\frac{es}{2}\sigma_{\mu\nu}F^{\mu\nu}\right).\end{aligned} \tag{33.76}$$

Again, this can be checked by differentiation. For a constant magnetic field, this is equivalent to Eq. (33.67).

The Euler–Heisenberg Lagrangian was first calculated by Heisenberg and his student Hans Euler by finding exact solutions to the Dirac equation in a constant $F_{\mu\nu}$ background [Euler and Heisenberg, 1936]. Our derivation of it, particularly the one in Appendix 33.A, is due to Schwinger [Schwinger, 1951].

## 33.4.1 Vacuum polarization

Expanding the unrenormalized Euler–Heisenberg Lagrangian, as in Eq. (33.72), we found two divergent terms which were removed with counterterms in Eq. (33.73). If we do not include these counterterms, the expansion gives

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{e^2}{8\pi^2}\int_0^\infty \frac{ds}{s}e^{is\varepsilon}e^{-sm^2}\left[\frac{1}{e^2s^2} + \frac{1}{6}F_{\mu\nu}^2\right] + \text{ finite}. \tag{33.77}$$

The first term in brackets is constant. It gives the vacuum energy density, as discussed in Section 33.2.3. The second term looks just like the tree-level QED kinetic term, $-\frac{1}{4}F_{\mu\nu}^2$. Keeping only this term (before renormalization), we have

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{6}F_{\mu\nu}^2\frac{e^2}{8\pi^2}\int_0^\infty \frac{ds}{s}e^{is\varepsilon}e^{-sm^2}. \tag{33.78}$$

This is UV divergent, from the $s \sim 0$ region. Regulating with a Lorentz-invariant UV cutoff $s_0$, we find

$$\begin{aligned}\mathcal{L}_{\mathrm{EH}} &= -\frac{1}{4}F_{\mu\nu}^2\left(1 + \frac{e^2}{12\pi^2}\int_{s_0}^\infty \frac{ds}{s}e^{is\varepsilon}e^{-sm^2}\right)\\ &= -\frac{1}{4}F_{\mu\nu}^2\left(1 - \frac{e^2}{12\pi^2}\ln\left(s_0m^2\right) + \text{ const}\right).\end{aligned} \tag{33.79}$$

This logarithmic dependence on the cutoff is exactly what we found from computing the full vacuum polarization graph in QED. As discussed in Chapter 23, UV divergences determine RGEs, and this one determines the leading order $\beta$-function coefficient. We can read off from the coefficient of the logarithm in Eq. (33.79) (as discussed in Chapter 23), that the $\beta$-function in QED at 1-loop is

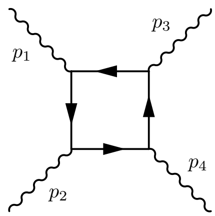
$$\beta(e) = \frac{e^3}{12\pi^2}, \tag{33.80}$$

which agrees with Eq. (16.73) (or Eq. (23.29)).

### 33.4.2 Light-by-light scattering

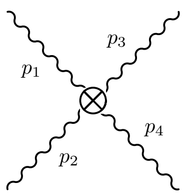
The original motivation of Heisenberg and Euler was to calculate the rate for photons to scatter off other photons. This problem was suggested to them by Otto Halpern and is sometimes called Halpern scattering. The relevant Feynman diagram is

$$i\mathcal{M} = \quad . \tag{33.81}$$




This is a difficult loop to compute directly, even with today's technology, much less with what Euler and Heisenberg knew in 1936. We can get the answer (in the limit of low-frequency light $\omega \ll m$) directly from the Euler–Heisenberg Lagrangian. The relevant term is the one to fourth order in $e$, which has the form $\frac{\alpha^2}{90}\frac{1}{m^4}\left[(F^2)^2 + \frac{7}{4}(F\tilde{F})^2\right]$. This term was computed first in a paper by Euler and Kockel [Euler and Kockel, 1935]. Using it for light-by-light scattering corresponds to a tree-level Feynman diagram of the form

$$i\mathcal{M} = \quad . \tag{33.82}$$




Note that our effective Lagrangian is only valid when $\partial_\mu F_{\alpha\beta} = 0$; thus we will only get the result to leading order in $\frac{p^2}{m^2}$. From the experimental point of view, this is enough, since light-by-light scattering of real on-shell photons has not yet been experimentally observed, at any frequency.

The matrix element is

$$\mathcal{M} = \frac{\alpha^2}{90}\frac{1}{m^4}\Big\{(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)(p^3_\alpha\epsilon^{3\star}_\beta - p^3_\beta\epsilon^{3\star}_\alpha)(p^4_\alpha\epsilon^{4\star}_\beta - p^4_\beta\epsilon^{4\star}_\alpha)$$
$$+\frac{7}{16}\left[\varepsilon^{\mu\nu\alpha\beta}(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha)\right] \times \left[\varepsilon^{\mu\nu\alpha\beta}(p^3_\mu\epsilon^{3\star}_\nu - p^3_\nu\epsilon^{3\star}_\mu)(p^4_\alpha\epsilon^{4\star}_\beta - p^4_\beta\epsilon^{4\star}_\alpha)\right]$$
$$+\text{permutations}\Big\}. \tag{33.83}$$

Summing over final polarizations and averaging over initial polarizations, the result is

$$\frac{1}{4}\sum_{\text{pols.}}\mathcal{M}^2 = \frac{1}{4}\frac{\alpha^4}{90^2}\frac{1}{m^8}2224(s^2t^2+s^2u^2+t^2u^2), \tag{33.84}$$

which leads to a cross section

$$\sigma_{\text{tot}} = \frac{973}{10125\pi}\alpha^4\frac{\omega^6}{m^8}. \tag{33.85}$$

This is the correct low-energy limit of the exact light-by-light scattering diagram. The exact result from the 1-loop graphs can be found in [Berestetsky *et al.*, 1982].

### 33.4.3 Schwinger pair production

Notice that the effective Lagrangian in Eq. (33.73) has singularities for certain values of the electromagnetic field. To see where the singularities are, we first consider the case with $\vec{B}$ and $\vec{E}$ parallel. Then,

$$F_{\mu\nu}^2 = 2(\vec{B}^2-\vec{E}^2) = 2\big(B^2-E^2\big)\,, \tag{33.86}$$

where $E=|\vec{E}|$ and $B=|\vec{B}|$, and

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = -4\vec{E}\cdot\vec{B} = -4EB, \tag{33.87}$$

and then, from Eq. (33.70),

$$X^2 = \frac{1}{2}(F_{\mu\nu}^2 - iF^{\mu\nu}\tilde{F}_{\mu\nu}) = (B+iE)^2. \tag{33.88}$$

Then the Euler–Heisenberg Lagrangian in Eq. (33.73) simplifies to

$$\begin{aligned}\mathcal{L}_{\text{EH}} = {}& \frac{1}{2}\left(E^2-B^2\right)\\ &-\frac{e^2}{8\pi^2}\int_0^\infty \frac{ds}{s}e^{i\varepsilon s}e^{-m^2 s}\left[EB\cot(esE)\coth(esB) - \frac{1}{e^2s^2} - \frac{B^2-E^2}{3}\right].\end{aligned} \tag{33.89}$$

Since $\coth(x)$ has no poles for $x>0$, the singularities are all associated with constant electric fields. Thus, we take the limit $B\to 0$, in which case the fact that we took $\vec{E}$ and $\vec{B}$ parallel is immaterial. From Eq. (33.89) we find

$$\mathcal{L}_{\text{EH}} = \frac{1}{2}E^2 - \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}e^{i\varepsilon s}e^{-sm^2}\left[eEs\cot(eEs) - 1 + \frac{1}{3}(esE)^2\right]. \tag{33.90}$$

In this form, we can see that the Euler–Heisenberg Lagrangian has poles for real $E$ when $s$ is equal to $s_n = \frac{n\pi}{eE}$ for $n=1,2,\ldots$ As we will now see, these poles indicate that strong electric fields can create electron–positron pairs, a process known as **Schwinger pair production** (although it was predicted first by Euler and Heisenberg).

How can electrons and positrons be produced from the Euler–Heisenberg Lagrangian, which has no electron field in it? They cannot. However, in a unitary quantum field theory, forward scattering rates are related to the sum over real production rates via the optical theorem. Recall from Section 24.1 that by the optical theorem (see Eq. (24.11))

$$\mathrm{Im}\mathcal{M}(A\to A)=\frac{1}{2}\sum_X d\Pi_{\mathrm{LIPS}}^X|\mathcal{M}(A\to X)|^2. \tag{33.91}$$

We can apply this theorem to QED in the situation where $|A\rangle$ corresponds to a coherent collection of photons describing a large electric field. In QED, the sum over states $|X\rangle$ includes states with on-shell electrons and positrons. Since QED is unitary, the optical theorem holds. In the Euler–Heisenberg Lagrangian the states $|A\rangle$ are the same states as in QED. Thus, if the calculation of $\mathcal{L}_{\mathrm{EH}}$ has been done correctly, the left-hand side of Eq. (33.91) should be unchanged, as one would expect from a matching calculation. The right-hand side of Eq. (33.91), on the other hand, cannot be the same as in full QED, since QED has electrons in it and the Euler–Heisenberg theory does not. Thus, what would be a unitary process in full QED now appears as a non-unitary process in the effective theory. Unfortunately, it is not easy to use Eq. (33.91) to calculate the pair-production rate, since one would have to sum over an infinite number of multi-particle states.

There is a nice shortcut, due to Schwinger, for evaluating the total pair-production rate. If there were no pair production, then the electric field state $|A\rangle$ would be constant in time. Thus $\langle A|S|A\rangle=1$ where $S$ is the $S$-matrix. Since in this case the action is constant, $S=e^{i\Gamma}$. Therefore, $|\langle A|\,e^{i\Gamma}\,|A\rangle|^2=\left|e^{i\Gamma}\right|^2$ measures the probability for something other than $A$ to be produced. In other words, $\left|e^{i\Gamma}\right|^2$ gives the probability that no pairs are produced over the time $T$ and volume $V$ of the experiment. We then have

$$\left|e^{i\Gamma}\right|^2=e^{i\Gamma}e^{-i\Gamma^\star}=e^{i(\Gamma-\Gamma^\star)}=e^{-2\mathrm{Im}[\Gamma]}=e^{-2VT\mathrm{Im}\mathcal{L}_{\mathrm{EH}}}, \tag{33.92}$$

where in the last step we use that, for given background fields, the Euler–Heisenberg Lagrangian is just a number. Thus $2\mathrm{Im}\mathcal{L}_{\mathrm{EH}}$ is the probability, per unit time and volume, that any number of pairs are created. This is the continuum field version of the optical theorem relation $\mathrm{Im}\mathcal{M}(A\to A)=m_A\Gamma_{\mathrm{tot}}$, where $\Gamma_{\mathrm{tot}}$ is the total decay rate of a single particle of mass $m_A$.

In order to calculate $\mathrm{Im}\mathcal{L}_{\mathrm{EH}}$ we note that the integrand in Eq. (33.71) has poles at $s_n=\frac{\pi}{eE}n$. There is no pole at $s=0$, as can be seen from expanding the integrand at small $s$. The imaginary part of this expression can be calculated using contour integration (Problem 33.3). The result is that[2]

$$2\mathrm{Im}(\mathcal{L}_{\mathrm{eff}})=\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{s_n^2}e^{-m^2s_n}=\frac{\alpha E^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(\frac{-n\pi m^2}{eE}\right). \tag{33.93}$$

Performing this sum, we find

$$\Gamma\big(E\to e^+e^-\ \text{pairs}\big)=\frac{\alpha E^2}{\pi^2}\,\mathrm{Li}_2\left(e^{-\frac{\pi m^2}{eE}}\right), \tag{33.94}$$

with $\mathrm{Li}_2(x)$ the dilogarithm function. This is the rate for Schwinger pair production in an external electric field.

[2] This sum also has an interpretation as a sum over instantons (see for example [Kim and Page, 2002]).

The rate for pair production is negligible until $E \gtrsim E_{\text{critical}} = \frac{m_e^2}{e} \approx 10^{18}$ volts/meter, which is an enormous field. As of this writing, Schwinger pair production in QED has still not been observed, since it is extremely difficult to get such fields in the lab. One might imagine, however, that such strong fields might be produced close to a particle with a very large charge, such as an atomic nucleus. The field around a nucleus is $E \sim \frac{e}{4\pi r^2} Z$. Now, the Euler–Heisenberg Lagrangian is only valid for fields that have wavelengths greater than $\frac{1}{m_e}$, so the best we can say is that pair production would begin for $Z$ large enough that $E_{\text{critical}} \sim \frac{e}{4\pi \left(m_e^{-2}\right)} Z$, which gives $Z = \frac{4\pi^2}{e^2} = \frac{1}{\alpha} \sim 137$. This result is sometimes invoked to explain why the periodic table has less than 137 elements![3]

### 33.4.4 Connection to perturbation theory

It is informative to consider which of the predictions we have derived from $\mathcal{L}_{\text{EH}}$ are equivalent to perturbative calculations in QED, and which are not.

We found that the Schwinger pair-production rate depended on $\exp(-\frac{\pi m^2}{eE})$. This dependence on $e$ indicates that pair production is a non-perturbative effect – you would never see pair production from constant electric fields at any fixed order in perturbative QED. Of course, you can get pair production in perturbation theory. But this would involve photon modes of frequencies larger than $m$. More precisely, one can show that [Itzykson and Zuber, 1980]

$$\Gamma(E \to e^+ e^-) = \frac{\alpha}{3} \int d^4 q \theta(q^2 - 4m^2) \left[\vec{E}(q^2)\right]^2 \sqrt{1 - \frac{4m^2}{q^2}} \left(1 + \frac{2m^2}{q^2}\right), \quad (33.95)$$

which vanishes when $\vec{E}$ is constant. The Schwinger pair-production rate is one of the very few analytic non-perturbative calculations in quantum field theory that give physical predictions.

Other results, such as the rate for light-by-light scattering, could be calculated in perturbative QED. Nevertheless, the Euler–Heisenberg Lagrangian efficiently encodes the result of many loop calculations all at once. It is worth discussing exactly what graphs are included in the Euler–Heisenberg Lagrangian, since this understanding will apply to similar effective actions in other contexts.

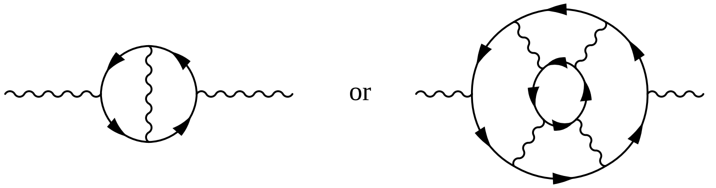
Recall our expression for the effective Lagrangian where the fermion is integrated out, Eq. (33.38),

$$\mathcal{L}_{\text{eff}}[A] = -\frac{1}{4} F_{\mu\nu}^2 + \frac{i}{2} \int \frac{ds}{s} e^{-ism^2} \langle x | e^{-i\not{D}^2 s} | x \rangle. \quad (33.96)$$

We have not assumed $F_{\mu\nu}$ is constant at this point, and in fact this effective action is *exact*. That is, since the Lagrangian was quadratic in $\psi$, this is a formal expression for the result of evaluating the path integral of $\psi$ completely. It does, however, correspond to only 1-loop

[3] This result actually follows more simply from dimensional analysis. The ground state of a hydrogen-like atom has energy $E_0 \sim -Z^2 \alpha^2 m_e$. To get pair production, a nucleus has to be able to capture an electron from the vacuum, emitting a positron into the continuum, so $E_0 \lesssim -m_e$ giving $Z \gtrsim \frac{1}{\alpha}$, up to order 1 factors, which we cannot get by dimensional analysis.

graphs, those in Eq. (33.40), since there is only a single propagator going from $x$ back to $x$ in proper time $s$. But how can this expression be exact if it does not include higher loops? Are graphs such as

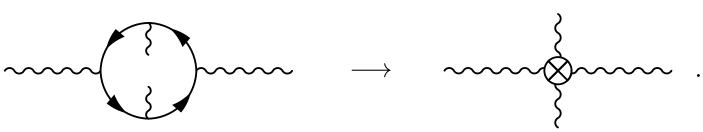



(33.97)

which have internal photon and/or fermion loops, included or not?

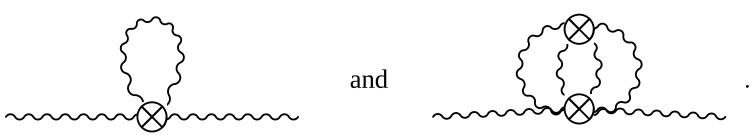
To answer this question, first recall that in the calculation of the effective action, and in the formal exact expression Eq. (33.38), the photon propagator plays no role. In fact, if we dropped the photon kinetic term from the original action, the only change in the effective action would be that the $-\frac{1}{4}F_{\mu\nu}^2$ term would be missing. Thus, neither of the graphs above are included in the effective action calculation, since both involve the photon propagator. On the other hand, since nothing is thrown out (assuming the effective action $\Gamma[A]$ is known exactly), any physical effect associated with these graphs must be reproducible within the effective theory. For example, these graphs in full QED contribute to the QED $\beta$-function, which has physical effects. The way the effective theory reproduces the physics of these loops is with its own loops involving effective vertices. Basically, the fermion loops are computed first, treating the photon lines as external, which generates new vertices. Then the photon lines coming off these vertices are sewn together in a loop amplitude using the photon propagator in the effective theory.

For example, to reproduce the physics of the first graph in Eq. (33.97), the relevant effective vertex can be determined by cutting through the intermediate photon and then contracting the fermion loop to a point:



(33.98)

The second graph in Eq. (33.97) involves this vertex, associated with the inner fermion loop, and a 6-point vertex associated with the outer fermion loop. The physics of the diagrams in Eq. (33.97) are then reproduced by connecting the legs in these effective vertices:




(33.99)

These graphs would reproduce the complete result from the graphs in Eq. (33.97), but we need the full $\mathcal{L}_{\text{eff}}[A]$ to compute them.

In the Euler–Heisenberg Lagrangian, we took $F_{\mu\nu}$ constant. Thus, the full physics of the loops in Eq. (33.97) is not reproduced by the Euler–Heisenberg Lagrangian alone. Only if we had the full effective Lagrangian, by evaluating $\Gamma[A]$ exactly, which would supplement the Euler–Heisenberg Lagrangian with additional terms depending on $\partial_\mu F_{\alpha\beta}$ (and give corrections at higher order in $\alpha$ to the terms without derivatives), would the full theory be reproduced. This exact $\Gamma[A]$ is not known.

Even at energies above $m_e$, the exact effective Lagrangian can be used. The electron still shows up as a pole in the scattering amplitude, as is clear already from Schwinger pair production in the constant $F_{\mu\nu}$ approximation. Thus, one can treat the electron like a bound state and calculate $S$-matrix elements for it. Of course, this is a terribly inefficient way to calculate electron production and scattering, since we already know the full theory. It is more efficient to use the UV completion of $\Gamma$, namely QED, which has a Lagrangian that is local and real.

## 33.5 Coupling to other currents

The effective action from integrating out $\psi$ can be generalized to the case where $\psi$ couples to other things besides $A_\mu$. In this way, we can calculate things such as the $\pi^0 \to \gamma\gamma$ rate, where $\pi^0$ is the neutral pion from QCD (see Chapter 28).

When $\psi$ couples to things other than $A_\mu$, the effective Lagrangian has more terms. Say we had

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{2}\phi(\Box + m_\phi^2)\phi - \frac{1}{2}\pi(\Box + m_\pi^2)\pi - eA_\mu\bar{\psi}\gamma^\mu\psi + \lambda\phi\bar{\psi}\psi + ig\pi\bar{\psi}\gamma^5\psi, \tag{33.100}$$

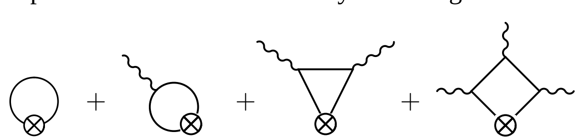
which has a scalar $\phi$ and a pseudoscalar $\pi$ in addition to the external field $A_\mu$. When we integrate out $\psi$, the effective Lagrangian (without $\psi$) will just contain the other fields coupled to the expectation value of the various $\psi$ bilinears in the background electromagnetic field, as in Section 33.2.2. That is,

$$\mathcal{L}_{\text{eff}}\,[A,\phi,\pi] = -\frac{1}{2}\phi(\Box + m_\phi^2)\phi - \frac{1}{2}\pi(\Box + m_\pi^2)\pi - eA_\mu J_A^\mu + \lambda\phi J_\phi + ig\pi J_\pi, \tag{33.101}$$

where

$$J_A^\mu = \langle A|\bar{\psi}\gamma^\mu\psi|A\rangle, \quad J_\phi = \langle A|\bar{\psi}\psi|A\rangle, \quad J_\pi = \langle A|\bar{\psi}\gamma^5\psi|A\rangle. \tag{33.102}$$

We sometimes call these field-dependent expectation values **classical currents**, since they are just classical functionals of background $A_\mu(x)$ fields. The calculation of these classical currents corresponds to the evaluation of Feynman diagrams such as

$J_\phi =$  $+ \cdots .$ (33.103)

Here, the $\otimes$ refers to insertions of the external current in the original theory, corresponding to an interaction with the scalar. The photon lines are the background electromagnetic fields.

For the scalar current,

$$\begin{aligned} J_\phi &= \langle A|\bar{\psi}(x)\psi(x)|A\rangle = -\mathrm{Tr}\left[\langle x|\hat{G}_A|x\rangle\right] \\ &= -\mathrm{Tr}\left[\int_0^\infty ds\, e^{-ism^2}\langle x|(\hat{\not{p}} - e\hat{\not{A}} + m)e^{i(\hat{\not{p}}-e\hat{\not{A}})^2 s}|x\rangle\right] \\ &= -4m\int_0^\infty ds\, e^{-ism^2}\langle x|e^{-i\hat{H}s}|x\rangle. \end{aligned} \tag{33.104}$$

You may notice that $J_\phi = -\frac{\partial}{\partial m}\mathcal{L}_{\text{eff}}[A]$, with $\mathcal{L}_{\text{eff}}[A]$ in Eq. (33.38), a result that is useful and not surprising, since the $\phi\bar{\psi}\psi$ interaction and the mass term $m\bar{\psi}\psi$ have the same form.

For the pseudoscalar current,

$$\begin{aligned} J_\pi &= \langle A|\bar{\psi}(x)\gamma^5\psi(x)|A\rangle = -\mathrm{Tr}\left[\langle x|\hat{G}_A\gamma^5|x\rangle\right] \\ &= -\mathrm{Tr}\left[\int_0^\infty ds\, e^{-ism^2}\langle x|(\hat{\not{p}} - e\hat{\not{A}} + m)e^{i(\hat{\not{p}}-e\hat{\not{A}})^2 s}\gamma^5|x\rangle\right] \\ &= -m\int_0^\infty ds\, e^{-ism^2}\mathrm{Tr}\left[\langle x|\gamma^5 e^{-i\hat{H}s}|x\rangle\right]. \end{aligned} \tag{33.105}$$

This current does not have a simple relation to $\mathcal{L}_{\text{eff}}[A]$, but as we will see, is not hard to compute.

### 33.5.1 Currents at low energy

Since the scalar current is $J_\phi = -\frac{\partial}{\partial m}\mathcal{L}_{\text{eff}}[A]$, for the case of constant electromagnetic fields, we can read the answer from the Euler–Heisenberg Lagrangian, although additional counterterms may be required. We find (hiding the counterterms)

$$\begin{aligned} J_\phi &= -\frac{e^2}{32\pi^2}\frac{\partial}{\partial m}\int_0^\infty \frac{ds}{s}e^{-m^2 s}\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}F^{\mu\nu}\tilde{F}_{\mu\nu} \\ &= \frac{e^2}{8\pi^2}\frac{\partial}{\partial m}\int_0^\infty \frac{ds}{s}e^{-m^2 s}\left[\frac{1}{e^2 s^2} + \frac{1}{6}F_{\mu\nu}^2 + \cdots\right] \\ &= -\frac{e^2}{4\pi^2}m\int_0^\infty ds e^{-m^2 s}\left[\frac{1}{e^2 s^2} + \frac{1}{6}F_{\mu\nu}^2 + \cdots\right]. \end{aligned} \tag{33.106}$$

The first term is infinite and can be removed with a renormalization of the bare term $\Lambda^3\phi$ in the Lagrangian. The second term is finite and gives

$$J_\phi = -\frac{\alpha}{6\pi}\frac{1}{m}\left(F_{\mu\nu}^2 + \cdots\right), \tag{33.107}$$

where the $\cdots$ are higher order in $e$.

For the pseudoscalar, we need

$$J_\pi = -m\int_0^\infty ds\, e^{-ism^2}\mathrm{Tr}[\gamma_5\langle x|e^{-i\hat{H}s}|x\rangle]. \tag{33.108}$$

Now, from Eq. (33.76),

$$\langle x|e^{-i\hat{H}s}|x\rangle = \langle x;0|x;s\rangle = -i\frac{e^2}{64\pi^2}\frac{F^{\mu\nu}\tilde{F}_{\mu\nu}}{\text{Im}\cos(esX)}\exp\Big(-i\frac{es}{2}\sigma_{\mu\nu}F^{\mu\nu}\Big). \quad (33.109)$$

and so

$$J_\pi = \frac{ie^2m}{64\pi^2}\int_0^\infty ds\, e^{-ism^2}\frac{F^{\mu\nu}\tilde{F}_{\mu\nu}}{\text{Im}\cos(esX)}\text{Tr}[\gamma_5 e^{-i\frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}s}]. \quad (33.110)$$

Since $\text{Tr}[\gamma_5] = \text{Tr}[\sigma_{\mu\nu}\gamma_5] = 0$, only terms with $\sigma_{\mu\nu}$ to an even power will survive. Using $(\sigma_{\mu\nu}F^{\mu\nu})^2 = 2F_{\mu\nu}^2 + 2i\gamma_5 F^{\mu\nu}\tilde{F}_{\mu\nu}$ we get

$$\text{Tr}[\gamma_5 e^{-i\frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}s}] = -4i\text{Im}\cos(esX). \quad (33.111)$$

And thus,

$$J_\pi = \frac{e^2m}{16\pi^2}\int_0^\infty ds\, e^{-ism^2}F^{\mu\nu}\tilde{F}_{\mu\nu} = -i\frac{\alpha}{4\pi m}F^{\mu\nu}\tilde{F}_{\mu\nu}. \quad (33.112)$$

Plugging $J_\phi$ and $J_\pi$ and the Euler–Heisenberg Lagrangian into Eq. (33.101) gives

$$\mathcal{L}_{\text{eff}}[A,\phi,\pi] = \mathcal{L}_{\text{EH}}[A] - \frac{1}{2}\phi(\Box + m_\phi^2)\phi + \frac{\lambda}{m}\phi\Big(-\frac{\alpha}{6\pi}F_{\mu\nu}^2 + \cdots\Big)$$
$$- \frac{1}{2}\pi(\Box + m_\pi^2)\pi + \frac{\alpha}{4\pi}\frac{g}{m}\pi F^{\mu\nu}\tilde{F}_{\mu\nu}. \quad (33.113)$$

Note that the $\pi$ coupling has just one term. The decay rates predicted from this effective Lagrangian are

$$\Gamma(\phi \to \gamma\gamma) = \frac{\alpha^2}{144\pi^3}\lambda^2\frac{m_\phi^3}{m^2}, \quad (33.114)$$

$$\Gamma(\pi \to \gamma\gamma) = \frac{\alpha^2}{64\pi^3}g^2\frac{m_\pi^3}{m^2}. \quad (33.115)$$

Not surprisingly, the pseudoscalar rate agrees exactly with Eq. (30.11). In this method of calculation, however, we gain additional insight into the associated anomaly.

## 33.5.2 Chiral anomaly

Connecting the $\pi \to \gamma\gamma$ rate to an anomalous symmetry is straightforward in the effective action language. Recall that the QED Lagrangian,

$$\mathcal{L} = \bar{\psi}\big(i\partial\!\!\!/ - eA\!\!\!/\big)\psi - m\bar{\psi}\psi, \quad (33.116)$$

is invariant under a vector symmetry, $\psi \to e^{i\alpha}\psi$, and, in the limit $m \to 0$, under a chiral symmetry, $\psi \to e^{i\gamma_5}\psi$. The associated Noether currents are $J^\mu = \bar{\psi}\gamma^\mu\psi$ and $J^{\mu5} = \bar{\psi}\gamma^\mu\gamma^5\psi$. By the equations of motion, the axial current satisfies

$$\partial_\mu J^{\mu5} = 2im\bar{\psi}\gamma^5\psi. \quad (33.117)$$

So the amount by which the axial current is not conserved is proportional to the fermion mass.

Now, we already calculated the expectation value of $\bar{\psi}\gamma^5\psi$ in the background electromagnetic field. In Eq. (33.112) we found $\langle A|\bar{\psi}\gamma^5\psi|A\rangle = i\frac{\alpha}{4\pi m}F^{\mu\nu}\tilde{F}_{\mu\nu}$. This is consistent with Eq. (33.117) only if

$$\langle A|\partial_\mu J^{\mu 5}|A\rangle = -\frac{\alpha}{2\pi}F^{\mu\nu}\tilde{F}_{\mu\nu}, \tag{33.118}$$

which agrees with Eq. (30.22).

## 33.6 Semi-classical and non-relativistic limits

The Schwinger proper-time method is not only useful for calculating loops using quantum mechanics, it also gives a new perspective on the semi-classical and non-relativistic limits of quantum field theory. In particular, it illustrates where the particles are hiding in the path integral. As we will see, Schwinger proper time lets us derive one-particle quantum mechanics as the low-energy limit of quantum field theory.

To begin, we return to the expression for the Green's function we derived above for a scalar particle in a background electromagnetic field, Eq. (33.22):

$$G_A(x,y) = \langle A|T\{\phi(x)\phi(y)\}|A\rangle = \int_0^\infty ds\, e^{-ism^2}\langle y|e^{-i\hat{H}s}|x\rangle, \tag{33.119}$$

with $\hat{H} = -(\hat{p} - eA(\hat{x}))^2$. This operator $\hat{H}$ is the Hamiltonian in a one-particle quantum mechanical system that generates translations in Schwinger proper time $s$. The function $G_A(x,y)$ is computed for constant electromagnetic fields in Appendix 33.A. In this section, we rewrite $G_A(x,y)$ in terms of a quantum mechanical path integral.

In quantum mechanics, the path integral gives the amplitude for a particle to propagate from $x^\mu$ to $y^\mu$ in time $s$ (see Section 14.2.2):

$$\langle y|e^{-i\hat{H}s}|x\rangle = \int_{z(0)=x}^{z(s)=y}\mathcal{D}z(\tau)\exp(i\int d\tau\, \mathcal{L}(z,\dot{z})), \tag{33.120}$$

where $\mathcal{L} = \hat{p}\dot{\hat{x}} - \hat{H}$ is the Legendre transform of the Hamiltonian. We would like to work out this Lagrangian in the case of a scalar in an electromagnetic field.

To simplify things, we first write $\hat{H} = -\hat{\Pi}^2$, where $\hat{\Pi}^\mu = \hat{p}^\mu - eA^\mu(\hat{x})$. The Heisenberg equations of motion for translation in $s$ are

$$\dot{\hat{x}}^\mu \equiv \frac{d\hat{x}^\mu}{ds} = i[\hat{H},\hat{x}^\mu] = i[-\hat{\Pi}^2,\hat{x}^\mu] = 2\hat{\Pi}^\mu, \tag{33.121}$$

where $[\hat{\Pi}^\mu,\hat{x}^\nu] = [\hat{p}^\mu,\hat{x}^\nu] = ig^{\mu\nu}$ has been used in the last step. So,

$$\mathcal{L} = \hat{p}^\mu\frac{\partial\hat{H}}{\partial p^\mu} - \hat{H} = -\hat{\Pi}^2 - 2eA^\mu\Pi^\mu = -\left(\frac{d\hat{x}^\mu}{2ds}\right)^2 - eA^\mu\frac{d\hat{x}^\mu}{ds}, \tag{33.122}$$

giving

$$\langle y|e^{-i\hat{H}s}|x\rangle = \int_{z(0)=x}^{z(s)=y}\mathcal{D}z(\tau)\exp\left(-i\int_0^s d\tau\left(\frac{dz^\mu}{2d\tau}\right)^2 - ie\int A_\mu(z)dz^\mu\right), \tag{33.123}$$

with the integral over $A_\mu$ a line integral along the path $z(s)$. So the Green's function is

$$G_A(x,y) = \int_0^\infty ds\, e^{-ism^2} \int_{z(0)=x}^{z(s)=y} \mathcal{D}z(\tau) \exp\left(-i\int_0^s d\tau \left(\frac{dz^\mu}{2d\tau}\right)^2 - ie\int A_\mu(z)dz^\mu\right). \tag{33.124}$$

This is an exact formal expression, only useful to the extent that we can solve for $z(\tau)$.

This world-line formulation was derived by a different method by Feynman [Feynman, 1950], although it had little application for many years. Interest in this approach was revived by Polyakov [Polyakov, 1981] in the context of string theory, and by Bern and Kosower [Bern and Kosower, 1992] who used it to develop an efficient way to compute loop diagrams in QCD.

## 33.6.1 Semi-classical limit

In the limit that a particle is very massive, loops involving that particle are suppressed. Thus, it should be possible to treat a massive particle classically and the radiation it produces quantum mechanically.

To take the large mass limit, we first rescale $s \to \frac{s}{m^2}$ and $\tau \to \frac{\tau}{m^2}$. This gives

$$G_A(x,y) = \frac{1}{m^2}\int_0^\infty ds\, e^{-is} \int_{z(0)=x}^{z(\frac{s}{m^2})=y} \mathcal{D}z(\tau) \times \exp\left(-i\int_0^s d\tau \left[m^2\left(\frac{dz^\mu}{2d\tau}\right)^2\right] - ie\int A_\mu(z)dz^\mu\right). \tag{33.125}$$

Now we see that, for large $m$, the $m^2(\frac{dz^\mu}{2d\tau})^2$ term completely dominates the path integral. Moreover, as $m \to \infty$, the action is dominated by the point of stationary phase, which is also the classical free-particle solution:

$$z^\mu(\tau) = x^\mu + v^\mu \tau, \tag{33.126}$$

where $v^\mu = \frac{y^\mu - x^\mu}{s}$ is the particle's velocity. So we get, rescaling $s \to sm^2$ back again, and plugging in the stationary phase solution,

$$G_A(x,y) = \int_0^\infty ds \exp\left(-i\left[sm^2 + \frac{(y-x)^2}{4s} + ev^\mu \int_0^s d\tau A_\mu(z(\tau))\right]\right). \tag{33.127}$$

The first two terms in the exponent are independent of $e$ and represent propagation of a free particle, similar to Eq. (33.11). The next term is equivalent to adding a term to the Lagrangian $\mathcal{L} = -eA_\mu J_c^\mu$, where $J_c^\mu$ is the source current from a classical massive particle moving at constant velocity:

$$J_c^\mu(x) = v^\mu \delta(x - v\tau). \tag{33.128}$$

In words, a heavy particle produces a gauge potential $A_\mu$ as if it is moving at a constant velocity.

This is the **semi-classical** limit. When a particle is heavy, the quantum field theory can be approximated by treating that particle as a classical source, but treating everything else quantum mechanically. You can study the fermion case in Problem 33.4.

## 33.6.2 Non-relativistic limit

In the non-relativistic limit, not only is the particle's mass assumed to be larger than the energy of typical photons, but the particle's velocity is also assumed to be much less than the speed of light. Define $\Delta t = y^0 - x^0$ and $\Delta x = |\vec{y} - \vec{x}|$. A particle moving slowly from $x^\mu$ to $y^\mu$ has $\Delta t \gg \Delta x$.

Separating out the time component, the 2-point function in Eq. (33.124) becomes

$$G_A(x,y) = \int_0^\infty ds \int_{z(0)=x}^{z(s)=y} \mathcal{D}z^0(\tau)\mathcal{D}\vec{z}(\tau) \times \exp\left(-i\int_0^s d\tau \left[\left(\frac{dz^0}{2d\tau}\right)^2 - \left(\frac{d\vec{z}}{2d\tau}\right)^2 + m^2\right] - ie\int A_\mu(z)dz^\mu\right). \tag{33.129}$$

The classical path that minimizes the action, from the large $m$ limit, has

$$z^0(\tau) = x^0 + \frac{\Delta t}{s}\tau. \tag{33.130}$$

We want to treat this time evolution classically, and leave the rest of the field fluctuations quantum mechanical. However, we can see that since both $(\frac{dz^0}{2d\tau})^2$ and $m^2$ are large, the stationary phase will have $\frac{\Delta t}{2s} \sim m$ and so $s \sim \frac{\Delta t}{2m}$. That is, the integral is dominated by the region near $z^0 = x^0 + 2m\tau$ and $s = \frac{\Delta t}{2m}$. To leading order in the expansion of $s$ and $z^0$ around their stationary-phase points, we then find

$$G_A(x,y) = \int_{z(0)=x}^{z(\frac{\Delta t}{2m})=y} \mathcal{D}\vec{z}(\tau) \exp\left(i\int_0^{\frac{\Delta t}{2m}} d\tau \left[\left(\frac{d\vec{z}}{2d\tau}\right)^2 - 2m^2\right] - ie\int A_\mu(z)dz^\mu\right). \tag{33.131}$$

Now we change variables to $\tau = \frac{t}{2m}$ to find

$$G_A(x,y) = \int_{z(0)=x}^{z(\Delta t)=y} \mathcal{D}\vec{z}(t) \exp\left(i\int_0^{\Delta t} dt \left[\frac{1}{2}m\left(\frac{d\vec{z}}{dt}\right)^2 - m\right] - ie\int A_\mu(z)dz^\mu\right). \tag{33.132}$$

This result is exactly the path integral expression in non-relativistic, first-quantized quantum mechanics with a potential $V = m$. We have just derived that the non-relativistic limit of quantum field theory is quantum mechanics!

## 33.A Schwinger's method

In this appendix, we explicitly calculate the 1-loop effective action for constant background electromagnetic fields $F_{\mu\nu}$ using Schwinger's original method [Schwinger, 1951]. This is an alternative way to calculate the Euler–Heisenberg Lagrangian than the sum over Landau levels method discussed in Section 33.4. This method, although a bit longer, is appealing because it avoids having to regulate the system in a box. It also produces a general expression for the propagator $G_A(x, y)$ of a particle in a constant background electromagnetic field.

Our starting point is the formula for the effective action in Eq. (33.38):

$$\mathcal{L}_{\text{eff}}(x) = -\frac{1}{4}F_{\mu\nu}^2(x) + \frac{i}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2}\text{Tr}\left[\langle x|e^{-i\hat{H}s}|x\rangle\right], \tag{33.A.133}$$

with $\hat{H} = -(\hat{p}^\mu - eA^\mu(\hat{x}))^2 + \frac{e}{2}F_{\mu\nu}(\hat{x})\sigma^{\mu\nu}$. We have dropped the $\varepsilon$ term, since we will not need it with this method. Here $A_\mu(\hat{x})$ is to be thought of as a classical gauge field configuration with position replaced by the operator $\hat{x}$. We would like to calculate $\mathcal{L}_{\text{eff}}(x)$ when $F_{\mu\nu}(\hat{x}) = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\hat{x})$ is constant. We begin by calculating $\langle y|e^{-i\hat{H}s}|x\rangle$. Once this is known, we will set $y = x$ and integrate over $s$ to get $\mathcal{L}_{\text{eff}}$.

### 33.A.1 Proper-time propagation

States such as $|x\rangle$ are eigenstates of an operator $\hat{x}^\mu$ in a first-quantized Hilbert space. The operators $\hat{x}^\mu$ are Schrödinger-picture operators. They are related to Heisenberg-picture operators by $\hat{x}^\mu(s) = e^{i\hat{H}s}\hat{x}^\mu e^{-i\hat{H}s}$. Using the definition $|x; s\rangle \equiv e^{-i\hat{H}s}|x\rangle$ we find

$$i\partial_s\langle y; 0|x; s\rangle = i\partial_s\langle y|e^{-i\hat{H}s}|x\rangle = \langle y|e^{-i\hat{H}s}\hat{H}|x\rangle. \tag{33.A.134}$$

Now,

$$\langle y|e^{-i\hat{H}s}\hat{x}^\mu(s) = \langle y|\hat{x}^\mu e^{-i\hat{H}s} = y^\mu\langle y|e^{-i\hat{H}s}, \tag{33.A.135}$$

and

$$\hat{x}^\mu(0)|x; 0\rangle = \hat{x}^\mu|x; 0\rangle = x^\mu|x; 0\rangle. \tag{33.A.136}$$

Thus, if we can write $\hat{H}$ in terms of $\hat{x}(0)$ and $\hat{x}(s)$ we can turn Eq. (33.A.134) into an ordinary differential equation whose solution gives $\langle y; 0|x; s\rangle$.

In quantum mechanics, the position and momentum operators satisfy $[\hat{x}, \hat{p}] = i$. In our 4D first-quantized setup we generalize this to

$$[\hat{x}^\mu(s), \hat{p}^\nu(s)] = -ig^{\mu\nu}, \tag{33.A.137}$$

with the commutation applying at the same proper time $s$. To simplify the form of the Hamiltonian, we introduce the operator $\hat{\Pi}^\mu = \hat{p}^\mu - eA^\mu(\hat{x})$. Then, assuming $F_{\mu\nu}$ is constant, we get

$$\left[\hat{x}^\mu(s), \hat{\Pi}^\nu(s)\right] = -ig^{\mu\nu}, \tag{33.A.138}$$

$$[\hat{\Pi}^\mu(s), \hat{\Pi}^\nu(s)] = -ieF^{\mu\nu}. \tag{33.A.139}$$

In terms of $\hat{\Pi}^\mu$, the Hamiltonian is

$$\hat{H}(s) = -\hat{\not{\Pi}}^2 = -\hat{\Pi}_\mu(s)\hat{\Pi}^\mu(s) + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}. \tag{33.A.140}$$

For simplicity, we will drop circumflexes on operators from now on. As a notational convenience, we will also replace $\mu$ and $\nu$ indices with boldface type. So the vectors $x^\mu$ and $\Pi^\mu$ are written as $\mathbf{x}$ and $\mathbf{\Pi}$, respectively, and the matrices $F^{\mu\nu}$ and $\sigma^{\mu\nu}$ are written as $\mathbf{F}$ and $\boldsymbol{\sigma}$ respectively. Then $\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F}) = \sigma_{\nu\mu}F^{\mu\nu} = -\sigma_{\mu\nu}F^{\mu\nu}$, with $\mathrm{tr}(\cdots)$ referring to a trace over $\mu$ and $\nu$ indices in this context.

In this notation, the evolution of $\Pi^\mu(s)$ generated by the Hamiltonian $H(s)$ through the Heisenberg equations of motion becomes

$$\frac{d\mathbf{\Pi}}{ds} = i[\hat{H}, \mathbf{\Pi}] = 2e\mathbf{F}\cdot\mathbf{\Pi}, \tag{33.A.141}$$

where we have used that since $\mathbf{F}$ is constant it commutes with all operators, including $\mathbf{\Pi}$. This equation is solved by $\mathbf{\Pi}(s) = e^{2es\mathbf{F}}\mathbf{\Pi}(0)$. Similarly,

$$\frac{d\mathbf{x}}{ds} = i[\hat{H}, \mathbf{x}] = 2\mathbf{\Pi}, \tag{33.A.142}$$

which gives

$$\mathbf{x}(s) = \mathbf{x}(0) + 2se^{es\mathbf{F}}\frac{\sinh(es\mathbf{F})}{se\mathbf{F}}\cdot\mathbf{\Pi}(0). \tag{33.A.143}$$

This solution is easy to check by differentiating. In the limit $\mathbf{A}\to 0$, $\mathbf{\Pi}\to\mathbf{p}$ and this becomes $\mathbf{x}(s) = \mathbf{x}(0) + 2s\mathbf{p}(0)$, which is consistent with the eigenstates of $\mathbf{x}(s)$ being those which evolve into position $x^\mu$ after a time $s$.

Thus we have

$$\mathbf{\Pi}(0) = e^{-es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}\cdot[\mathbf{x}(s) - \mathbf{x}(0)], \tag{33.A.144}$$

$$\mathbf{\Pi}(s) = e^{es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}\cdot[\mathbf{x}(s) - \mathbf{x}(0)]. \tag{33.A.145}$$

The Hamiltonian then becomes

$$\hat{H} = -\mathbf{\Pi}(s)\cdot\mathbf{\Pi}(s) - \frac{e}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F}) = -[\mathbf{x}(s) - \mathbf{x}(0)]\,\mathbf{K}[\mathbf{x}(s) - \mathbf{x}(0)] - \frac{e}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F}), \tag{33.A.146}$$

with $\mathbf{K} \equiv \frac{e^2\mathbf{F}^2}{4\sinh^2(e\mathbf{F}s)}$. Note that $K_{\mu\nu} = K_{\nu\mu}$.

To evaluate $\langle y|e^{-i\hat{H}s}\hat{H}|x\rangle$ in Eq. (33.A.134) using $\hat{H}$, it is helpful first to rewrite $\hat{H}$ so that $\mathbf{x}(s)$ is on the left and $\mathbf{x}(0)$ is on the right. This is not hard:

$$\mathbf{\Pi}(s)\cdot\mathbf{\Pi}(s) = \mathbf{x}(s)\mathbf{K}\mathbf{x}(s) - 2\mathbf{x}(s)\mathbf{K}\mathbf{x}(0) + \mathbf{x}(0)\mathbf{K}\mathbf{x}(0) + K_{\mu\nu}[x^\mu(s), x^\nu(0)]. \tag{33.A.147}$$

Now,

$$\begin{aligned} K_{\mu\nu}[x^\mu(s), x^\nu(0)] &= -\mathrm{tr}\left\{\mathbf{K}\left[\mathbf{x}(0), \mathbf{x}(0) + 2e^{es\mathbf{F}}\frac{\sinh(es\mathbf{F})}{e\mathbf{F}}\cdot\mathbf{\Pi}(0)\right]\right\} \\ &= \frac{i}{2}\mathrm{tr}[e\mathbf{F} + e\mathbf{F}\coth(es\mathbf{F})]. \end{aligned} \tag{33.A.148}$$

So, since $\mathrm{tr}[\mathbf{F}] = 0$, we have

$$\hat{H} = -\mathbf{x}(s)\mathbf{K}\mathbf{x}(s) + 2\mathbf{x}(s)\mathbf{K}\mathbf{x}(0) - \mathbf{x}(0)\mathbf{K}\mathbf{x}(0) - \frac{i}{2}\mathrm{tr}[e\mathbf{F}\coth(es\mathbf{F})] - \frac{e}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})\,. \tag{33.A.149}$$

In this canonical form, $\hat{H}$ can be evaluated in position eigenstates.

Equation (33.A.134) becomes

$$i\partial_s\langle y;0|x;s\rangle = -\left\{(\mathbf{y}-\mathbf{x})\frac{e^2\mathbf{F}^2}{4\sinh^2(es\mathbf{F})}(\mathbf{y}-\mathbf{x}) + \frac{i}{2}\mathrm{tr}[e\mathbf{F}\coth(es\mathbf{F})] + \frac{e}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})\right\}\langle y;0|x;s\rangle, \tag{33.A.150}$$

where $\mathbf{x} = x^\mu$ and $\mathbf{y} = y^\mu$ are position vectors, not operators anymore. This is just a differential equation. The general solution is

$$\langle y;0|x;s\rangle = C(x,y)\exp\left\{i(\mathbf{y}-\mathbf{x})\frac{e\mathbf{F}}{4}\coth(es\mathbf{F})(\mathbf{y}-\mathbf{x}) - \frac{1}{2}\mathrm{tr}\ln\left[\frac{\sinh(es\mathbf{F})}{e\mathbf{F}}\right] + i\frac{es}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})\right\} \tag{33.A.151}$$

This can be checked by differentiation and holds for any $C(x,y)$.

To determine $C(x,y)$, we use the additional information that

$$\left(i\frac{\partial}{\partial\mathbf{x}} - e\mathbf{A}\right)\langle y;0|x;s\rangle = \langle y;0|e^{-i\hat{H}s}\mathbf{\Pi}(0)|x;s\rangle = e^{-es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}(\mathbf{y}-\mathbf{x})\langle y;0|x;s\rangle, \tag{33.A.152}$$

and similarly

$$\left(-i\frac{\partial}{\partial\mathbf{y}} - e\mathbf{A}\right)\langle y;0|x;s\rangle = e^{es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}(\mathbf{y}-\mathbf{x})\,\langle y;0|x;s\rangle\,. \tag{33.A.153}$$

Plugging in our general solution, we find

$$\left[i\frac{\partial}{\partial\mathbf{x}} - e\mathbf{A} - \frac{e}{2}\mathbf{F}(\mathbf{x}-\mathbf{y})\right]C(x,y) = 0, \tag{33.A.154}$$

and

$$\left[-i\frac{\partial}{\partial\mathbf{y}} - e\mathbf{A} - \frac{e}{2}\mathbf{F}(\mathbf{x}-\mathbf{y})\right]C(x,y) = 0. \tag{33.A.155}$$

The solution is

$$C(x,y) = C\exp\left[ie\int_x^y dz^\mu\Big(A_\mu(z) + \frac{1}{2}F_{\mu\nu}(z^\nu - y^\nu)\Big)\right]. \tag{33.A.156}$$

This line integral is independent of path since the integrand has zero curl. The constant $C$ can be fixed by demanding that the result reduce to the free theory as $A \to 0$. The final result is

$$\langle y;0|x;s\rangle = \frac{-i}{16\pi^2 s^2}\exp\left[ie\int_x^y dz^\mu\Big(A_\mu(z)+\frac{1}{2}F_{\mu\nu}(z^\nu-y^\nu)\Big)\right]$$
$$\times\exp\left[i(\mathbf{y}-\mathbf{x})\frac{e\mathbf{F}}{4}\coth(es\mathbf{F})(\mathbf{y}-\mathbf{x})+i\frac{es}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})-\frac{1}{2}\mathrm{tr}\ln\left[\frac{\sinh(es\mathbf{F})}{es\mathbf{F}}\right]\right], \tag{33.A.157}$$

which is manifestly gauge invariant. Taking $A\to 0$ reproduces Eq. (33.11), which confirms the normalization.

Equation (33.A.157) is more generally useful than just for the calculation of the Euler–Heisenberg Lagrangian. The special case when $x=y$ is quoted in Eq. (33.76) and used for the calculation of the $\pi^0\to\gamma\gamma$ rate in Section 33.5.1.

## 33.A.2 Effective Lagrangian

Now that we have the proper-time Hamiltonian, we are a small step away from the Euler–Heisenberg Lagrangian. We need to calculate

$$\mathcal{L}_{\mathrm{EH}}(x) = -\frac{1}{4}F_{\mu\nu}^2(x)+\frac{i}{2}\int_0^\infty\frac{ds}{s}e^{-ism^2}\mathrm{Tr}\left\{\langle x|e^{-i\hat{H}s}|x\rangle\right\}$$
$$= -\frac{1}{4}F_{\mu\nu}^2(x)+\frac{1}{32\pi^2}\mathrm{Tr}\left\{\int_0^\infty ds\frac{1}{s^3}\exp\left[-ism^2+i\frac{es}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})-\frac{1}{2}\mathrm{tr}\ln\left[\frac{\sinh(es\mathbf{F})}{es\mathbf{F}}\right]\right]\right\}, \tag{33.A.158}$$

where $\mathrm{Tr}$ is the Dirac trace and $\mathrm{tr}$ contracts $\mu$ and $\nu$ as above.

Now, recall from Eq. (30.65) that

$$[\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})]^2 = -2\mathrm{tr}\big(\mathbf{F}^2\big)-2i\gamma_5\mathrm{tr}(\mathbf{F}\widetilde{\mathbf{F}}) = 8(\mathcal{F}-i\gamma_5\mathcal{G}), \tag{33.A.159}$$

where $\tilde{F}^{\mu\nu}\equiv\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and

$$\mathcal{F}\equiv\frac{1}{4}F_{\mu\nu}^2=\frac{1}{2}(\vec{B}^2-\vec{E}^2), \tag{33.A.160}$$
$$\mathcal{G}\equiv-\frac{1}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}=\vec{E}\cdot\vec{B}. \tag{33.A.161}$$

Then, since $\gamma_5$ has eigenvalues $\pm1$, the Dirac eigenvalues of $\mathrm{Tr}(\boldsymbol{\sigma}\mathbf{F})$ are

$$\lambda_i^{\boldsymbol{\sigma}\mathbf{F}}=\pm\sqrt{8(\mathcal{F}\pm i\mathcal{G})}, \tag{33.A.162}$$

with all four sign combinations possible. So,

$$\mathrm{Tr}\left[e^{i\frac{es}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})}\right] = 2\cos\left[es\sqrt{2(\mathcal{F}+i\mathcal{G})}\right]+2\cos\left[es\sqrt{2(\mathcal{F}-i\mathcal{G})}\right]$$
$$= 4\mathrm{Re}\cos[esX]\,, \tag{33.A.163}$$

where

$$X\equiv\sqrt{\frac{1}{2}F_{\mu\nu}^2-\frac{i}{2}F^{\mu\nu}\tilde{F}_{\mu\nu}}=\sqrt{2\,(\mathcal{F}+i\mathcal{G})}=\sqrt{(\vec{B}+i\vec{E})^2}. \tag{33.A.164}$$

Next we need

$$\frac{1}{2}\text{tr}\ln\left[\frac{\sinh(e\mathbf{F}s)}{es\mathbf{F}}\right] = \ln\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}, \tag{33.A.165}$$

where $\lambda_i$ are the four eigenvalues of $\frac{\sinh(e\mathbf{F}s)}{es\mathbf{F}}$. These eigenvalues are determined from the eigenvalues of a constant $F_{\mu\nu}$, which are (see Problem 33.5)

$$\lambda_i^{\mathbf{F}} = \pm\frac{i}{\sqrt{2}}\left[\sqrt{\mathcal{F}+i\mathcal{G}} \pm \sqrt{\mathcal{F}-i\mathcal{G}}\right], \tag{33.A.166}$$

with all four possible sign choices. After some simplification the result is

$$\exp\left\{-\frac{1}{2}\text{tr}\ln\left[\frac{\sinh(e\mathbf{F}s)}{es\mathbf{F}}\right]\right\} = -\frac{(es)^2\mathcal{G}}{\text{Im}\cos(esX)}. \tag{33.A.167}$$

Putting everything together, we find

$$\mathcal{L}_{\text{EH}}(x) = -\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{32\pi^2}\int_0^\infty ds\frac{1}{s}e^{-im^2 s}\frac{\text{Re}\cos(esX)}{\text{Im}\cos(esX)}F^{\mu\nu}\tilde{F}_{\mu\nu}, \tag{33.A.168}$$

which is the final answer for the unrenormalized Euler–Heisenberg effective Lagrangian, in agreement with Eq. (33.71).

## Problems

**33.1** Complete the calculation of the Euler–Heisenberg Lagrangian using Landau levels in an arbitrary $F_{\mu\nu}$. Show that for an electric field $B \to iE$ is justified. Also show that the result for a general electromagnetic field is given by Eq. (33.71).

**33.2** Calculate light-by-light scattering using helicity spinors.

**33.3** Calculate the contour integral to derive the pair-production rate Eq. (33.94) from Eq. (33.93). It is helpful to first expand the integration limits to $\int_{-\infty}^{\infty} ds$, then deform the contour to pick up the poles.

**33.4** Repeat the analysis in Section 33.6.1 for a fermion. Show that in the non-relativistic limit, the spin is irrelevant.

**33.5** Show that the eigenvalues of $F_{\mu\nu}$ are given by Eq. (33.A.166).